257B - Aspects of Fukaya categories

* Course website: math.stanford.edu/~ganatra/math257B/
* Note takers for each lecture appreciated
* No class on Monday April 4th
* Weeks 2-[redacted] WF 13h30-14h50, after MW 13h30-14h50.

1) Definitions / motivations / a first glimpse:

* (X^{2n}, ω) symplectic manifold, $L^n \subseteq X^{2n}$ Lagrangian if $\omega|_L = 0$.
* $H : X \to \mathbb{R}$ Hamiltonian $\rightsquigarrow X_H$ ham vector field ($\iota_{X_H}\omega = dH$)

$\rightsquigarrow \phi_H^s$ time-s flow

We allow time-dependent hamiltonians $H : (S^1 \text{ or } [0,1]) \times X \to \mathbb{R}$.

1)a. Lagrangian Floer homology:

impressionistically, it associates (in nice cases) to a pair (L_0, L_1) of Lagrangians a group $HF^*(L_0, L_1)$ satisfying formally:

* "categorifies intersection #": $\chi(HF^*(L_0, L_1)) = L_0 \cdot L_1$ ← smooth topology
* Ham. isotopy invariant: $HF^*(\phi_H^1 L_0, L_1) \cong HF^*(L_0, L_1) \cong HF^*(L_0, \phi_H^1 L_1)$.
* If $L_0 \pitchfork L_1$, arrange that $HF^*(L_0, L_1) = H^*(CF^*(L_0, L_1), \delta)$
  ↳ gen. by intersection points

$\Rightarrow \phi_H L_0 \cap L_1 \geq \operatorname{rk} HF^*(L_0, L_1) \geq L_0 \cdot L_1$

So, $\operatorname{rk} HF^*$ gives a refined lower bound for Lagrangian intersections.

(i) ex: $L := S^1 \subseteq \mathbb{C}$ (unit). Note $\exists\, H : \mathbb{C} \to \mathbb{R} : (x+iy) \mapsto 3y$, with $X_H = -3\partial_x$ such that $\phi_H(L) \cap L = \emptyset$: "L is displaceable".

So, if $HF^*(L, L)$ existed and satisfied the properties above, we would have $HF^*(L, L) = 0$.

(ii) Say $\pi_2(X, L) = 0$. Floer proved that $HF^*(L, L)$ is defined, and $= H^*_{sing}(L)$. By (i), it can not be displaceable.

(iii) "Everything is a Lagrangian" (Weinstein).

Famous conjecture of Arnold: if $H : S^1 \times X \to \mathbb{R}$ generic (fixed points of ϕ^1_H isolated), then $\# \mathrm{Fix}(\phi_H) \geq \mathrm{rk}\, H^*(X)$ (often $\geq \chi(X)$, by Lefschetz fixed point)

<u>Observe:</u> given any $\phi : X \to X$ symplectomorphism, $\Gamma_\phi \subseteq \underset{(X, -\omega)}{\bar{X}} \times X$ is Lagrangian, and $\Delta \cap \Gamma_\phi \xleftrightarrow{1:1} \mathrm{Fix}(\phi)$.

Arnold's conjecture would follow by showing $\mathrm{rk}\, HF(\Delta, \Gamma_\phi) = \mathrm{rk}\, H^*(\Delta) = \mathrm{rk}\, H^*(X)$, but there are more direct methods.

(iv) There are cases where we can define $CF^*(L_0, L_1) \circlearrowleft \delta$, but $\delta^2 \neq 0$.

<u>ex:</u>  . Following FOOO, we say that (L_0, L_1) is <u>obstructed</u>, or L is obstructed if $L_0 = L_1 = L$.

<u>Note:</u> we'll eventually need to clarify some issues:

$HF^*(L_0, L_1)$ — grading? field? — ~~Often~~ At first, $\mathbb{Z}_2$ and $\mathbb{Z}_2$, but by choosing extra data on L_0 and L_1, we can often lift it to $\mathbb{Z}$ and $\mathbb{C}$, or $\mathbb{Z}$ and Λ (Novikov field, needed for convergence issues). 

To define δ, we will study spaces of J-holomorphic discs in Π, for some (auxilary) almost complex structure J on M.

1) b. <u>Fukaya categories</u> keep track of relationship between $HF^*(L_0, L_1)$ for varying L_0, L_1.

[Donaldson]: he observed that there is a "composition"

$[\mu^2] : HF^*(L_1, L_2) \otimes HF^*(L_0, L_1) \to HF^*(L_0, L_2)$ using the ambiant geometry.

Thus, L_i are objects of a category: the Donaldson-Fukaya category H^0F.

<u>Objects</u>: $L_i \subseteq X$ Lagrangians (unobstructed)

<u>Morphisms</u>: $\mathrm{Hom}(L_i, L_j) = HF^*(L_i, L_j)$ (check: identity morphisms)

Unfortunately, this is unsufficient for many purposes, e.g. for building / iterating LES in $HF^*(-,-)$. Instead, we work at the chain level: $\hom(L_0, L_1) := CF^*(L_0, L_1) \circlearrowleft \delta = \mu^1$, and we have $\mu^2 : CF^*(L_1, L_2) \otimes CF^*(L_0, L_1) \to CF^*(L_0, L_2)$.

<u>Problem</u>: μ^2 is not associative :-(

Instead, the associator $\mu^2(-, \mu^2(-,-)) - \mu^2(\mu^2(-,-), -)$ $= \mu^3(\mu^1(-), -, -) + \dots$, i.e. it is chain homotopic to zero, for a chain homotopy μ^3.

[Fukaya]: there is a hierarchy $\mu^k : CF^*(L_{k-1}, L_k) \otimes \dots \otimes CF^*(L_0, L_1) \to CF^*(L_0, L_k)$ satisfying $0 = \sum_{i,j} (-1)^{\dots} \mu^d(x_d, \dots, x_{i+j+1}, \mu^j(x_{i+j}, \dots, x_{i+1}), x_i, \dots, x_1)$ $\forall d$.

The first three are $(\mu^1)^2 = 0$, μ^2 chain map, μ^3 as above.

<u>Claim</u>: $(\mathcal{F}(M), \{\mu^i\})$ A_∞ category is a quasi-isomorphism invariant.

$\mathcal{F}(M)$ is the right setting to talk about relations between Lagrangians (such as exact triangles, etc). Or rather, take the split-closed derived category $D^\pi \mathcal{F}(M)$, whose objects are $\{L_0 \to L_1 \to \dots \to L_k\}$.

<u>ex</u>: $L_2 \simeq \{L_0 \to L_1\}$ means LES $\forall K$:

$$\begin{tikzcd} HF^*(L_2, K) \arrow[rr] & & HF^*(L_0, K) \arrow[dl] \\ & HF^*(L_1, K) \arrow[ul] & \end{tikzcd}$$

<u>Rem</u>: $D^\pi \mathcal{F}(M)$ is relevant to mirror symmetry, a series of duality between symplectic geometry of (X, ω) and complex geometry of $(\check{X}, J)$, discovered in string theory.

(4)

"mirror quartic"

(*) ex: [CdOGP '91] $\{n_d\}$ on X^S $\rightsquigarrow$ $\{N_d^B\}$ on $(\check{X}, J)$

degree d curves — quintic 3-fold $\subseteq \mathbb{P}^4$ — "period integrals": $\int \Omega \wedge D_a D_b D_c \Omega$

Conjecture: [Kontsevich] HMS: for such $(X, \check{X})$,

$$D^\pi \mathbb{F}(X, \omega) \cong D^b \operatorname{Coh}(\check{X}, J)$$

Lagrangians — coherent sheaves, built out of hol. vector bundles and holomorphic submanifolds.

and moreover Kontsevich conjectured that this recovers (*).

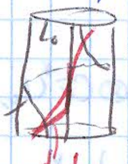
1) c. Flavors of Fukaya categories & their relations

(useful in computing ordinary Fukaya categories!)

If M is non-compact, we might try to put non-compact $L \subseteq M$ in $\mathbb{F}(M)$; there are then several choices for how to define the category:

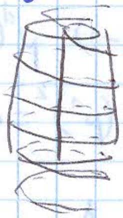
(i) "small asymmetric Ham. perturbations near ∞": $\hom(L_0, L_1) := CF^*(\phi_\varepsilon L_0, L_1)$

positive pushoff



$\rightsquigarrow$ "infinitesimal Fukaya category"

$\mathcal{W}(M)$

(ii) "arbitrarily large perturbations":  $\rightsquigarrow$ "wrapped Fukaya category".

(iii) Fukaya categories of singularities: $\mathbb{F}(E, W)$ for $W: E \to \mathbb{C}$ symplectic fibration with singular points (ex: Lefschetz fibrations, ie holomorphic Morse fibrations). This W {allows specifies the non-compactness direction.

Rem: * $D^\pi \mathbb{F}(E, W)$ arises in HMS as mirror to $D^b \operatorname{Coh}(\check{X})$, for $\check{X}$ non Calabi-Yau: $c_1(\check{X}) \neq 0$.

* $D^\pi \mathbb{F}(E, W)$ is an invariant of the singularities of W.

ex: $\mathbb{C}^n$, W polynomial.

Meta-statements: [Seidel, Abouzaid-Seidel, Abouzaid-Ganatra]

1) In many cases, $\mathcal{F}(E,w)$ is easier to compute than $\mathcal{F}(E)$ (functors, exact sequences, etc).

2) $M := w^{-1}(p)$, there are relations between $\mathcal{F}(E,w)$ and $\mathcal{F}(M)$, allowing us to
- compute $\mathcal{F}(M)$
- study monodromy $\mu : M \circlearrowleft M$

3) Can use $\mathcal{F}(E,w)$ and $\mathcal{F}(E)$ as stepping stones to understand $\mathcal{F}(E)$ and $\mathcal{W}(E)$ (the wrapped Fukaya category).