Meta-statements: [Seidel, Abouzaid-Seidel, Abouzaid-Ganatra]

1) In many cases, $\mathcal{F}(E,W)$ is easier to compute than $\mathcal{F}(E)$ (functors, exact sequences, etc).

2) $M := W^{-1}(p)$, there are relations between $\mathcal{F}(E,W)$ and $\mathcal{F}(M)$, allowing us to - compute $\mathcal{F}(M)$
- study monodromy $\mu : M \circlearrowleft M$

3) Can use $\mathcal{F}(E,W)$ and $\mathcal{F}(E)$ as stepping stones to understand $\mathcal{F}(E)$ and $\mathcal{W}(E)$ (the wrapped Fukaya category).

30/03/16

J-holomorphic curves & Lagrangian Floer homology, I:

Recall: (X^{an}, ω) symplectique

Definitions: an almost complex structure J is $J \in End(TM)$ s.t. $J^2 = -\mathbb{1}$.

J is compatible with ω if $\omega(-, J-)$ is a metric

J is tamed by ω if $\omega(v, Jv) > 0$ if $v \neq 0$ ($\Rightarrow \frac{\omega(v,Jw)+\omega(Jv,w)}{2}$ is a metric).
↳ open condition.

Rem: the entire theory we'll develop can be done for tame a.c.s. It is sometimes handy, as there exists more tame a.c.s.

ex: for (X,ω) a Kähler manifold, $J : TX \circlearrowleft$ induced is an integrable structure.

J-holomorphic curve: fix (X, J) almost complex manifold, (Σ, j) a Riemann surface (possibly with ∂, and with standard complex structure).

Definition: $u : \Sigma \to X$ is J-holomorphic if $du \circ j = J \circ du$
$\Leftrightarrow \bar{\partial}_J u := \frac{1}{2}(du + J \circ du \circ j) = 0$. In local coordinates on Σ, $s+it$, this is equivalent to the usual Cauchy-Riemann equations $\partial_t u = J \partial_s u$.

⊛

Given a metric g (and a fixed one on Σ), get a metric on $\mathrm{Maps}(T\Sigma, u^*TX)$, so we can define the <u>energy</u> of such a map: $E(u) := \int_\Sigma |du|^2$.

Proposition (identity energy): if ω is symplectic, J compatible, $g := \omega(-, J-)$ and u is J-holomorphic, then

$$E(u) = \int_\Sigma u^*\omega \qquad \text{(exercise)}.$$

↳ only depends on homotopy class of u ⇒ a priori controlled.

⊛ Rem: we might impose Lagrangian boundary conditions, namely $u|_{\partial\Sigma}$ maps into Lagrangian submanifolds.

Towards Lagrangian Floer homology:

Let $L_0^n, L_1^n \subseteq (X, \omega)$ Lagrangian submanifolds.

The Floer homology, formally, will be the Morse homology theory for a symplectic action functional on $\mathcal{P}(L_0, L_1) = \{\gamma : [0,1] \to X \mid \gamma(0) \in L_0, \gamma(1) \in L_1\}$.

↳ "$\mathcal{A} : \mathcal{P}(L_0, L_1) \to \mathbb{R}$".

Recap: if $f : M^k \to \mathbb{R}$ is a Morse function on M (smooth, finite dimensional), we get $HM^*(f) = H^*(CM^*(f), \delta)$ where $CM^*(f) = \bigoplus_{p \in \mathrm{crit}(f)} \mathbb{C}\langle p \rangle$ and δ counts flowlines of ∇f (w.r.t. some g) between critical points p and q. A flowline is a map $\gamma : \mathbb{R} \to M$ st $\dot\gamma = \nabla f$, $\lim_{s\to-\infty} \gamma(s) = p$, $\lim_{s\to+\infty} \gamma(s) = q$.

Actually, it turns out that in general, only $d\mathcal{A}$ is well-defined (which is good enough to deal with $\nabla\mathcal{A}$). We have

$$\mathcal{A} : \tilde{\mathcal{P}}(L_0, L_1) \to \mathbb{R} : (\gamma, [u]) \mapsto \int u^*\omega$$

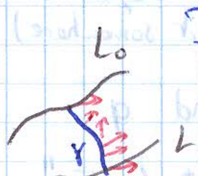
↳ univ. cover: elements are $(\gamma, [u])$, where $u : [0,1]^2 \to X$ is a path in $\mathcal{P}(L_0, L_1)$ from $*$ (a base point) to γ.

↳ we assume $\mathcal{P}(L_0, L_1)$ is connected.

Rem: there are cases in which $\mathcal{A} : \mathcal{P}(L_0, L_1) \to \mathbb{R}$ is defined.

ex: (X, ω) exact if $\omega = d\lambda$ for some λ (need X non-compact for that; eg $\mathbb{C}^n$ with ω_{std}). Fixing λ, we say that $L \subseteq X$ is exact if $\lambda|_L = df$, for $f : L \to \mathbb{R}$. We often, in this case, fix f as before, and call (L, f) an exact Lagrangian for (X, λ).

Exercise: if (L_0, f_0) and (L_1, f_1) are exact Lagrangians in (X, λ) exact symplectic, then $\mathcal{A}(\gamma, [u]) = \tilde{\mathcal{A}}(\gamma) := \int_{[0,1]} \gamma^* \lambda \pm f_1(\gamma(1)) \mp f_0(\gamma(0))$.



Define $T_\gamma \mathcal{P}(L_0, L_1) = \{$vector fields on γ keeping the end points on L_0 and $L_1\}$

$= \{$vector fields v on $\gamma^* TX$ s.t. $v(0) \in T_{\gamma(0)} L_0,\ v(1) \in T_{\gamma(1)} L_1\}$.

Given $(\gamma, [u])$, note that using variational calculus,

$$\underset{\in T_\gamma \mathcal{P}(L_0, L_1)}{d\mathcal{A}(\gamma) \cdot v} = \int_{[0,1]} \omega(\dot\gamma, v)\, dt \overset{\text{fix } J}{=} \int_{[0,1]} g(J\dot\gamma, v)\, dt = \langle J\dot\gamma, v \rangle_{L^2}$$

$\hookrightarrow$ using g

(Point here is that a choice of J induces a metric on $T_\gamma \mathcal{P}(L_0, L_1)$ via $\langle v_0, v_1 \rangle_{L^2} := \int_{[0,1]} \omega(v_0, Jv_1)\, dt$.)

$\Rightarrow \nabla \mathcal{A}_\gamma = -J\dot\gamma$

Rem: $d\mathcal{A}(\gamma)$ could have a priori depended on $[u]$, but we see that it doesn't.

Hence:
- Critical points of $\mathcal{A}$ $\longleftrightarrow$ $\dot\gamma = 0$, i.e. constant paths
  $\longleftrightarrow$ intersection points $p \in L_0 \cap L_1$.
- Gradient trajectories = "J-hol maps": $\gamma_s : \mathbb{R}_s \to \mathcal{P}(L_0, L_1)$ with $\frac{\partial \gamma}{\partial s} = -J\dot\gamma = -J\partial_t \gamma$ (which is the Cauchy-Riemann equation).

It ends up being rather difficult to define ∞-dimensional ($\mathcal{P}(L_0, L_1)$ is infinite dimensional) Morse theory directly using variational methods, even in this instance. For instance,

(1) In Morse theory, an important notion is the $\text{index}_f(p)$, which is

the "number of negative eigenvalues of Hess(f) at p". At a critical point of $\mathcal{A}$, the index in this sense is ∞ (there are ∞ many $+$ and $-$ eigenvalues).

(2) The gradient flow of $\nabla\mathcal{A}$ is not well-defined: $\nabla\mathcal{A}_\gamma$ is generally not even tangent to L_0 and L_1.

[Floer]: we can still make sense of

- gradient flow lines, thought of as solutions to a PDE, instead of the gradient flow equation for $\mathcal{A}$. (so we think of the entire flow line, instead of taking a point and moving it somewhere)
- a relative index, depending on a choice of path between p and q (or homotopy class thereof): "# eigenvalves that cross from $\pm$ to $\mp$".

Actual setup:

Say $L_0 \pitchfork L_1$, for $L_0, L_1 \subseteq (X,\omega)$ Lagrangians.

Define Λ to be our base field, and $T \in \Lambda$ a choice of element. In our setting, most generally, Λ = Novikov field over $\mathbb{k} = \mathbb{C}$, $= \{\sum a_i T^{\lambda_i} \mid a_i \in \mathbb{k}, \lambda_i \in \mathbb{R}, \lambda_i \to \infty\}$

and T is just T.

In nice cases, we might have $\Lambda = \mathbb{C}((T))$ with T the formal variable, or even $\Lambda = \mathbb{C}$ with $T = 1$.

Floer complex: $CF^*(L_0,L_1) := \Lambda^{|L_0 \cap L_1|}$ the free Λ-module generated by $L_0 \cap L_1$.

Goal: define $\partial : CF^*(L_0,L_1) \circlearrowleft$ by counting J-holomorphic discs.

Specifically, look at $u : (\mathbb{R}_s \times [0,1]_t, j) \to X$ equipped with an a.c.s. J, such that

$$(*)\quad \begin{cases} 1)\ \bar\partial_J u = \frac{1}{2}(du + J \circ du \circ j) = 0 \\ 2)\ u(s,0) \in L_0,\quad u(s,1) \in L_1 \\ 3)\ \lim_{s\to+\infty} u(s,t) = p \in L_0 \cap L_1 \text{ (cst path at } p) \\ 4)\ \lim_{s\to-\infty} u(s,t) = q \end{cases}$$

5) $E(u) \overset{\text{prop}}{=} \int u^*\omega = \iint_{\mathbb{R}\times[0,1]} \left|\frac{\partial u}{\partial s}\right|^2 < \infty.$

Such a u is called a "J-holomorphic strip"

Proposition: 1), 2) and 5) imply that at $\pm\infty$, u (exponentially) converges to some elements of $L_0 \cap L_1$.

Note: $\mathbb{R} \times [0,1] \underset{\text{bihol}}{\simeq} \mathbb{D}^2 \setminus \{\pm 1\}$, so we can think of these as maps from

Define $\mathcal{M}(p, q, \underset{\substack{\cap \\ \pi_2(M; L_0, L_1)}}{[u]}, J) = \{u \text{ solutions to } (*) \text{ in homotopy class } [u]\}$.

and $\mathcal{M}(p,q,J) = \coprod_{\beta \in \pi_2(\ldots)} \mathcal{M}(p,q,\beta,J)$.

We want to do something like "count rigid elements of $\mathcal{M}(p,q,J)$", but there are not enough interesting rigid objects: given any non-constant u solving $(*)$ and any $\lambda \in \mathbb{R}$, $\tilde{u}(s,t) := u(s+\lambda, t)$ also solves $(*)$. So we get a free $\mathbb{R}$ action on non-constant ~~paths~~ parts of $\mathcal{M}(p,q,J)$.

Instead, we want to define, for $p \in CF^*(L_0, L_1)$ the generator corresponding to $p \in L_0 \cap L_1$,

$$\partial(p) = \sum_{\substack{q \in L_0 \cap L_1 \\ u \in \mathcal{M}(p,q,J)/\mathbb{R} \text{ rigid}}} \operatorname{sgn}(u) \cdot T^{\omega(u)} \cdot q$$

$$= \sum_{q,\, \beta \in \pi_2(\ldots)} \underset{\substack{\uparrow \\ \text{signed}}}{\#} \left(\frac{\mathcal{M}(p,q,\beta,J)}{\mathbb{R}}\right) \cdot T^{\int \beta^* \omega} \cdot q$$

Issues:
* what does it mean to count the number of elements solving a PDE on an ∞-dimensional manifold?

work for nice L_0 and L_1

* why is that count finite, well-defined, etc? (for generic J)
* why does $\partial^2 = 0$? (for generic J)
* why $HF^*(L_0, L_1) = H^*(CF^*(L_0, L_1), \partial)$ independent of choice of generic J?
* how to define $\operatorname{sgn}(u)$?

The analytic theory eventually will say that $\mathcal{M}(p,q,\beta,J)/\mathbb{R}$ is an orientable manifold of a certain finite dimension (when 0-dimensional, a choice of orientation is a sign at each point).

We want to see "$\mathcal{M}$ solves a Fredholm problem". We can write $\mathcal{M}(p,q,\beta,J) = \bar\partial_J^{-1}(0)$ where $\bar\partial_J : \mathcal{B} \to \mathcal{E}$ is a section of an infinite dimensional vector bundle over an infinite dimensional manifold

$$\mathcal{B} := C^\infty(\Sigma = \mathbb{R}\times[0,1], M; L_0, L_1, p, q, \beta).$$

The fiber of $\mathcal{E}$ over $u \in \mathcal{B}$ is $\mathcal{E}_u := C^\infty(\mathbb{R}\times[0,1], \Omega^{0,1}_\Sigma \otimes u^*TX)$

suppressing boundary / asymptotic conditions that should be here

It turns out that $\bar\partial_J$ is an elliptic operator, which for our purposes means that on suitable Sobolev completions

$$\mathcal{B}_{k+1,p} := W^{k+1,p}(\Sigma, X, \text{asymptotics/boundary}) \quad \text{"Banach manifold"}$$

$$\mathcal{E}^{k,p}_u := W^{k,p}(\Sigma, \Omega^{0,1}_\Sigma \otimes u^*TX, \ldots) \quad \text{"Banach vector bundle"}$$

we have $\bar\partial_J : \mathcal{B}_{k+1,p} \to \mathcal{E}^{k,p}$, and its linearization $D^u_{\bar\partial_J}$ is Fredholm, meaning it has finite dimensional kernel and cokernel

Definition: $u \in \mathcal{M}(p,q)$ is regular if $D^u_{\bar\partial_J}$ is onto (cokernel is empty).

In this case, a version of usual transversality theory tells us that $\bar\partial_J^{-1}(0)$ is a (locally near u) finite dimensional manifold with dimension $= \dim\ker D^u_{\bar\partial_J}$