01/04/16 <u>Last time</u>: $L_0, L_1 \subseteq (X^{2n}, \omega)$ Lagrangian submanifolds, $L_0 \pitchfork L_1$.

Fix $\Lambda$ field and $T \in \Lambda$:

- realistically, $\Lambda = \{ \sum a_i T^{\lambda_i} \mid \lambda_i \to \infty, \lambda_i \in \mathbb{R}, a_i \in k = \mathbb{C} \}$, $T$ the formal variable
- dream (+ some nice situations): $\Lambda = \mathbb{C}$, $T = 1$.

We tentitavely defined, based on the idea of Morse theory for $\mathcal{A} : \overset{(\sim)}{\mathcal{P}}(L_0, L_1) \to \mathbb{R}$, $CF^*(L_0, L_1) = \Lambda^{|L_0 \cap L_1|}$ and, wrt some almost complex structure $J$ on $X$, a "differential"

$$\partial p = \sum_{\substack{q,\, u \in \mathcal{M}(p,q,J)/\mathbb{R} \\ \text{rigid}}} T^{\omega(u)} \cdot \underset{\uparrow \pm 1 \text{ (imagine } \Lambda = \mathbb{Z}_2 \text{, so } +1)}{\mathrm{sgn}(u)} \cdot q ,$$

where $\mathcal{M}(p,q,J) = \coprod_\beta \mathcal{M}(p,q,\beta,J)$, $\beta \in \pi_2(M; L_0, L_1, p, q)$

and $\mathcal{M}(p,q,\beta,J) = \left\{ \begin{array}{l} u : (\mathbb{R} \times [0,1], j) \to (X, J) \text{ in class } \beta \\ u(s,i) \in L_i \text{ for } i \in \{0,1\} \\ \lim_{s\to+\infty} u(s,t) = p, \quad \lim_{s\to-\infty} u(s,t) = q \\ \bar\partial_J u = \frac{1}{2}(du + J \circ du \circ j) = 0 \\ E(u) = \int u^*\omega < \infty \end{array} \right\}$

<u>Want</u>: (a) For generic $J$, each $\mathcal{M}(p,q,\beta,J)$ is a finite-dimensional manifold, with a fixed "expected dimension" $=: \mathrm{ind}_{\bar\partial_J}(\beta)$.

(b) $\mathcal{M}(p,q,\beta,J)$ compact(ifiable) with the desired "boundary strata"

(c) If the above hold, $\delta$ is well-defined and $\delta^2 = 0$.

Let's talk about (a). Last time, we observed we could express $\mathcal{M}(p,q,\beta,J) = \bar\partial_J^{-1}(0)$ where $\bar\partial_J$ is a section of an infinite-dimensional vector bundle $\begin{array}{c}\mathcal{E} \\ \downarrow \pi \\ \mathcal{B}\end{array}$, for

$\mathcal{B} = C^\infty(\mathbb{R} \times [0,1], X, L_0, L_1, p, q, \beta)$

$\mathcal{E}_u = C^\infty(\underbrace{\mathbb{R} \times [0,1]}_{=: S}, \Omega_S^{0,1} \otimes u^*TX)$.

Toy model from finite dimensional differential topology: if $f : M^m \xrightarrow{C^\infty} N^n$ submersion at $p \in M$ (ie $Df_p : T_pM \twoheadrightarrow T_{f(p)}N$), the implicit function theorem tells us that $\underset{f(p)}{f^{-1}(q)}$ near $p$ has a smooth manifold structure, of dimension $m-n$.

Somewhat closer, $V$ rank $k$ vector bundle, want $s^{-1}(0)$ to be a smooth manifold of the "right dimension" $(m-k)$ $\Leftrightarrow$ $s(M) \pitchfork$ 0-section in $V$.

$$\begin{tikzcd} V \arrow[d, "\pi"] \\ M^m \arrow[u, bend left, "s"] \end{tikzcd}$$

The implicit function theorem applies at $p \in s^{-1}(0)$, provided we check $d^v s : T_p M \longrightarrow T^{vert}_{s(p)} V := \ker d\pi$ is surjective.

Moreover, $T_p(s^{-1}(0)) = \ker d^v s$.

Even if $d^v s$ is not surjective at $p$, we can determine an "expected dimension" as $\operatorname{ind}(d^v_p s) := \operatorname{rk} \ker d^v_p s - \operatorname{rk} \operatorname{coker} d^v_p s$ (it is $m-k$ here, independent of $p$).

For us, $M$ and $V$ are infinite dimensional, but in the <u>nicest possible</u> <u>way</u>:

**Theorem [Floer]:** if $L_0 \pitchfork L_1$, after extending $\bar\partial_J$ to a suitable Sobolev completion, solving $\bar\partial_J u = 0$ is a Fredholm problem, meaning that we have

$$\begin{tikzcd} \mathcal{E}^{k-1,p} \arrow[d, "\pi"] & \text{(Banach bundle)}, \\ \mathcal{B}^{k,p} \arrow[u, bend left, "\bar\partial_J"] & \text{(Banach manifold)} \end{tikzcd}$$

such that the linearization (vertical part of $D(\bar\partial_J)$) at $u \in \bar\partial_J^{-1}(0)$

$$D^u_{\bar\partial_J} : W^{k,p}(S, u^*TX, u^*TL_0, u^*TL_1) \longrightarrow W^{k-1,p}(S, u^*TX)$$

is Fredholm, i.e.
- the image is closed
- $\operatorname{ind}(D^u_{\bar\partial_J}) = \operatorname{rk}\ker(D^u_{\bar\partial_J}) - \operatorname{rk}\operatorname{coker}(D^u_{\bar\partial_J}) < \infty$.

We say that $u$ is regular is $D^u_{\bar\partial_J}$ is surjective. In this case, an infinite-dimensional version of the implicit function theorem says that near $u$, $M(p,q)$ is a finite dimensional manifold of dimension $= \mathrm{ind}(D^u_{\bar\partial_J}) = \mathrm{rk}\ker(D^u_{\bar\partial_J})$ and $T_u\mathcal{M} = \ker(D^u_{\bar\partial_J})$.

We'll see, as in the finite-dimensional case, $\mathrm{ind}(D^u_{\bar\partial_J})$ is independant of $u$, only depends on the topological data $[u]$ (cf Atiyah-Singer index theorem).

Unfortunately, a given $J$ may not be regular $\forall u$.

Theorem 2 [Floer]: In nice cases $(*)$ (such as when $\pi_2(M, L_i) = 0$ and $\pi_2(M) = 0$), there is a set of second Baire category (in particular, dense) of compatible $J$ such that $D^u_{\bar\partial_J}$ is onto $\forall u$.

$(*)$: when every $u$ (& their limits) is simple (somewhere injective).
In general, one might need "more advanced Fredholm differential topology".

Idea of proof: if $\mathcal{J}$ = space of compatible almost complex structures, we have an extended $\bar\partial_J : \mathcal{B} \times \mathcal{J} \xrightarrow{\bar\partial_{ex}} \mathcal{E} : (u, J) \mapsto \bar\partial_J u$.
Floer proved that under the hypothesis $(*)$ (and after Sobolev completing, etc) $\bar\partial_{ex}$ is a submersion. The IFT $\Rightarrow \mathcal{M}^{ex} := \bar\partial_{ex}^{-1}(0) \subseteq \mathcal{B} \times \mathcal{J}$ is a Banach submanifold (note it is not finite dimensional, because $\mathcal{J}$ is infinite dimensional).

Consider
$$\begin{array}{ccc} \mathcal{M}^{ex} & \subset & \mathcal{B} \times \mathcal{J} \\ \downarrow \pi' & & \downarrow \pi \\ \mathcal{J} & & \mathcal{J} \end{array}$$
Note $(\pi')^{-1}(\tilde J) = \mathcal{M}(p, q, \tilde J)$.

The $\infty$-dim Sard-Smale theorem $\Rightarrow$ the regular values of $\pi'$ are dense. But at regular values, $(\pi')^{-1}(\tilde J) = \mathcal{M}(p, q, \tilde J)$ is a submanifold of the right dimension.

What is $\mathrm{ind}(D^u_{\partial_J})$? Depends only on $[u] := \mathrm{ind}(\beta^{\in \pi_2(\ldots)})$

Detour: Maslov index:

Let $\Lambda(n)$ be the Lagrangian grassmannian in $\mathbb{C}^n$:
$$\Lambda(n) = \{ L^n \subseteq \mathbb{C}^n \text{ linear Lagrangian subspace} \}.$$
We have $\Lambda(n) \cong U(n)/O(n)$, and hence $H^1(\Lambda(n); \mathbb{Z}) \cong \mathbb{Z}$. The generator $\mu$ is called the Maslov class. → classifying map of $\mu$.

We have $\pi_1(\Lambda(n)) \cong \mathbb{Z}$, explicitely $U(n)/O(n) \xrightarrow{\det^2} S^1$ is a $\pi_1$-isomorphism, and moreover $\langle \mu, \gamma \text{ a loop of } \{L_t\} \rangle$ is the winding number of $\det^2 \circ \gamma$.

[Arnold]: geometric interpretation of the Maslov class.

Let $\Lambda_1 := \{\text{Lagr. planes that are not transverse to } \mathbb{R}^n \subseteq \mathbb{C}^n\} \subseteq \Lambda$ be the "Maslov cycle".

↳ or some other fixed Lagrangian subspace

$$\langle \mu, \gamma : S^1 \to \Lambda(n) \rangle = \gamma \cdot \Lambda_1$$
↳ signed intersection number

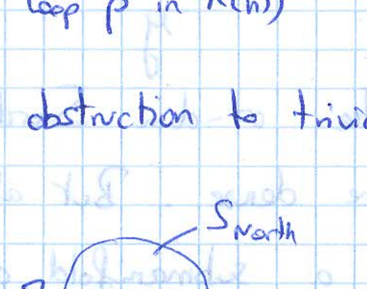
The relevance to index theory comes from the following toy example.

Observation: given a trivial $\mathbb{C}^n$-bundle $\begin{matrix} E \\ \downarrow \\ \mathbb{D}^2 \end{matrix}$ equipped with a Lagrangian sub-bundle $\begin{matrix} F \\ \downarrow \\ \partial \mathbb{D}^2 = S^1 \end{matrix} \subseteq E|_{S^1}$, (this data is the same as a loop $\rho$ in $\Lambda(n)$)

the Maslov index $\mu(\rho) =: \mu(E, F)$ is the obstruction to trivializing $F \subseteq E|_{S^1}$.

"Relative Chern class".

Exercise: $\begin{matrix} E \cong \mathbb{C}^n \times S^2 \\ \downarrow \\ S^2 \supseteq \Gamma \text{ equatorial } S^1 \end{matrix}$




If I pick a Lagrangian subbundle $F$ of $E|_\Gamma$, then
$$\mu(E_{North}, F) + \mu(E_{South}, F) = 2 \cdot c_1(E)[S^2].$$

The first index theorem (not quite the desired one) involving $\mu$ is:

<u>Theorem</u> (Riemann-Roch for surfaces with boundary) Let $(\Sigma, j)$ be a Riemann surface with $\partial\Sigma = S_1 \cup \dots \cup S_k$.
Let $\begin{matrix} E \\ \downarrow \\ (\Sigma, j) \end{matrix}$ be a holomorphic vector bundle with Lagrangian sub-bundles $F_i \subseteq E|_{S_i}$.
Then, the index of $\bar\partial_J : C^\infty(\Sigma, E)_{F_i} \to C^\infty(\Sigma, \Omega^{0,1}\Sigma \otimes E)$ is $(rk_{\mathbb{C}} E)\cdot \chi(\Sigma) + \sum \mu(E, F_i)$

↳ wrt a fixed trivialization of $E$.
ok because $E$ retracts on $V S^1$.

<u>Rem</u> if $\partial\Sigma = \phi$, get $(rk_{\mathbb{C}} E)\cdot\chi(\Sigma) + \langle 2c_1(E), [\Sigma]\rangle$.

<u>For Floer trajectories</u>: we need a definition of Maslov index for a pair of paths in $\Lambda(n)$.

Let $L_0, L_1(t)$ for $t \in [0,1]$ Lagrangian subspaces in $\Lambda(n)$ with $L_1(0) \pitchfork L_0$ and $L_1(1) \pitchfork L_0$. The Maslov index $(L_0, L_1(t))$ is the # of times where $L_1(t)$ fails to be transverse to $L_0$, counted with signs and multiplicities.

(ie if $L_0 = \mathbb{R}^n$, then index $L_1(t) \cdot \Lambda_1$)

( ⚠ we have $L_1(t)$ just a path now, not a loop. So the subspace fixed that we choose to define $\Lambda_1$ matters)

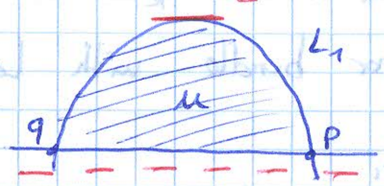
<u>ex</u>: $L_0 = \mathbb{R}^n \subseteq \mathbb{C}^n$, $L_1$ path: $(e^{i\pi\theta_1 t}\mathbb{R}) \times (e^{i\pi\theta_2 t}\mathbb{R}) \times \dots \times (e^{i\pi\theta_n t}\mathbb{R})$

- if all $\theta_i \neq 0, 1$, $L_0 \pitchfork L_1$ at $0, 1$
- if $\theta_i$ distinct, positive $\in (0,1)$, one can see that $\mu(L_0, L_1(t)) = n$.

Now, given a strip $u: \overbrace{\mathbb{R} \times [0,1]}^{=:S} \to (X, L_0, L_1)$, trivialize $u^*TX \cong S \times \mathbb{C}^n$. Get $u^*TL_0$, $u^*TL_1$ paths of Lagrangians along $\mathbb{R}\times\{0\}$, $\mathbb{R}\times\{1\}$. We can actually further trivialize, so that $u^* TL_0$ remains constant $\cong (\mathbb{R}\times\{0\})\times\mathbb{R}^n \subset (\mathbb{R}\times\{0\})\times\mathbb{C}^n$.

Then: $\mathrm{ind}([u]) :=$ Maslov index of the path $TL_1$ relative to $TL_0$ as one goes from $p$ to $q$.

ex:




fails to be transverse to $TL_0$, just once, positively. $\Rightarrow \mathrm{ind}([u]) = 1$.

(have here to choose a generator of $\pi_1(S^1)$ to talk of positivity).

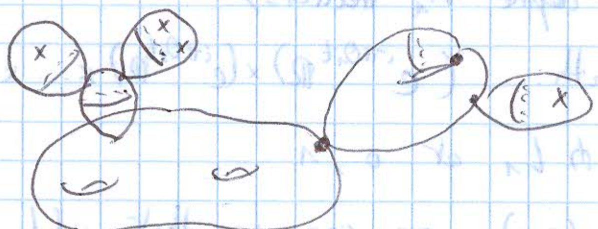
Let us now address (b): the compactness. We need to know that the numbers we are counting are finite. This is one of the key places where we use $\omega$ and the energy considerations.

Let's do a statement first for arbitrary $J$-hol curves.

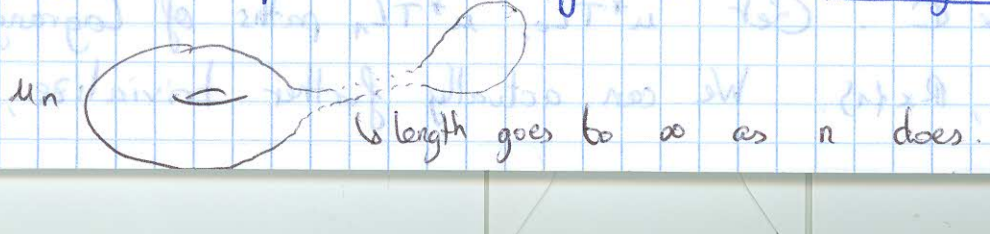
**Theorem (Gromov compactness)** Let $u_n : \Sigma_n \to X$ a sequence of $J$-holomorphic curves (assume for a minute $\partial\Sigma_n = \emptyset$), maybe with marked points, with energy a priori bounded independent of $n$:

$$E(u_n) = \int_{\Sigma_n} u_n^*\omega = \langle [\omega], u_{n*}[\Sigma_n]\rangle < K.$$

Then, $\exists$ subsequence converging to a <u>stable</u> $J$-holomorphic map $u_\infty : \Sigma_\infty \to X$, ie $\Sigma_\infty = \bigcup$ (nodal Riemann surfaces) with all marked points and nodes distinct in the domain (if they come together, create a constant bubble to keep them separated).



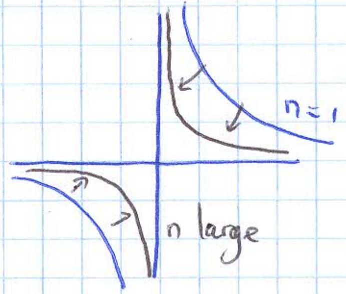
Phenomenon: besides possible degenerations of domain $(\Sigma_n, j_n)$, the main new phenomenon so far is the <u>bubbling of spheres</u>.



ex: $u_n : S^2 = \mathbb{CP}^1 = \mathbb{C} \cup \{\infty\} \longrightarrow \mathbb{CP}^1 \times \mathbb{CP}^1$

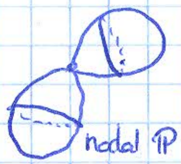
$$(x_0 : x_1) \longmapsto ((x_0 : x_1), (nx_1 : x_0)).$$

In an affine chart $x = \frac{x_1}{x_0}$, $\quad x \mapsto (x, \frac{1}{nx})$ (extended to $0, \infty$).
$\qquad \mathbb{C} \to \mathbb{C} \times \mathbb{C}$




Away from $0$, uniform convergence to $x \mapsto (x, 0)$ so the limit seems to be one line, but if we reparametrize $\tilde{x} = nx$, we get $\tilde{x} \mapsto (\tilde{x}/n, \frac{1}{\tilde{x}})$. This converges uniformly away from $\tilde{x} = \infty$ to $\tilde{x} \mapsto (0, \frac{1}{\tilde{x}})$, so we get the second coordinate axis.

So the limit of $u_n$ is a map




$\longrightarrow \mathbb{CP}^1 \times \mathbb{CP}^1$.

<u>Next time</u>: we'll see that there can be disc bubbles too.