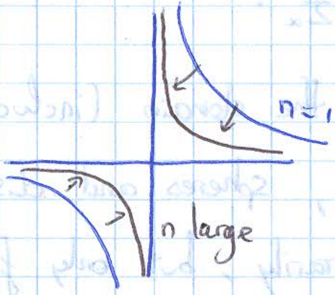
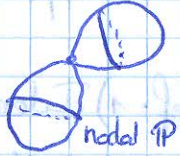
ex: $u_n : S^2 = \mathbb{CP}^1 = \mathbb{C} \cup \{\infty\} \longrightarrow \mathbb{CP}^1 \times \mathbb{CP}^1$

$$(x_0 : x_1) \longmapsto ((x_0 : x_1),\ (nx_1 : x_0)).$$

In an affine chart $x = \frac{x_1}{x_0}$, $\quad x \mapsto (x, \frac{1}{nx})$ (extended to $0, \infty$).
$\qquad \mathbb{C} \to \mathbb{C} \times \mathbb{C}$




Away from $0$, uniform convergence to $x \mapsto (x, 0)$ so the limit seems to be one line, but if we reparametrize $\tilde{x} = nx$, we get $\tilde{x} \mapsto (\tilde{x}/n, \frac{1}{\tilde{x}})$. This converges uniformly away from $\tilde{x} = \infty$ to $\tilde{x} \mapsto (0, \frac{1}{\tilde{x}})$, so we get the second coordinate axis.

So the limit of $u_n$ is a map  $\longrightarrow \mathbb{CP}^1 \times \mathbb{CP}^1$.


<u>Next time</u>: we'll see that there can be disc bubbles too.

06/04/16 <u>Last time</u>: statement of Gromov's compactness theorem.

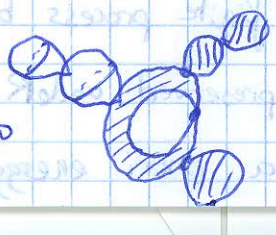
We will restate it, this time for $\Sigma$ with boundary.

<u>Theorem</u>: Suppose $\Sigma_n = (\Sigma, j_n)$ has boundary components $\partial_1\Sigma, \ldots, \partial_k\Sigma$ and is equipped with "marked points" on $\Sigma$ (possibly on $\partial\Sigma$). Suppose $u_n : \Sigma_n \to (X, J_n, \omega)$ is a sequence of J-hol. maps satisfying some Lagrangian boundary conditions $u_n(\partial_i\Sigma) \subseteq L_i$ Lagrangians in $X$, (⊗) and with energy $E(u_n) < K$ independent of $n$.

Then $\exists\ \Sigma_\infty$ "nodal surface" and a subsequence of $u_n$'s converging to a <u>stable</u> J-holomorphic map $u_\infty : \Sigma_\infty \to X$ (with Lagrangian boundary conditions, etc, as before).

⊗ rather, each component of $\partial_i\Sigma \setminus$ boundary marked points is sent to a Lagrangian in $X$.

ex: 

could converge to 

Convergence means, in the $C^\infty$ (or $C^l$) topology, for each component $\Sigma_\alpha$ of $\Sigma_\infty$, $\exists$ subregion $\Sigma_\alpha^n$ of $\Sigma_n$ and an automorphism $\phi_\alpha$ of $\Sigma_\alpha^n$ with $u_n \circ \phi_\alpha \xrightarrow{\text{on compact subsets}} u_\infty|_{\Sigma_\alpha}$.

<u>Phenomena:</u> in addition to degenerations of the domain (including boundary pinching  ), spheres and discs can bubble off. This can happen essentially arbitrarily, but only finitely many times.

<u>Rem:</u> the theorem requires $X$ compact, or $u_n: \Sigma \to C \subseteq X$ compact subset.

<u>Ideas of proof:</u> 1) Identify bubbling regions where $\sup|du_n| \to \infty$. Away from these points, standard analytic estimates and elliptic bootstrapping imply convergence on compact subsets to a $J$-hol map.

2) Say we have a sequence $z_n^\circ \in \Sigma_n$ where $|du_n| \to \infty$ <u>interior</u> points. In these regions, rescale $v_n(z) := u_n(z_n^\circ + \varepsilon_n z)$ for $\varepsilon_n \to 0$ suitable so that derivative doesn't go to $\infty$ anymore. Then, a subsequence of $v_n(z)$'s converge to a $J$-hol map $\mathbb{C} \to X$. By a <u>removal of singularities property</u> for $J$-hol maps, this extends to a map $\mathbb{CP}^1 = \mathbb{C} \cup \{\infty\} \to X$, a <u>sphere bubble</u>.

3) If instead $|du_n| \to \infty$ for $z_n^\circ$ a collection of <u>boundary points</u>, the same argument produces $\mathbb{H} \to X$ which compactifies to $\mathbb{D}^2 = \mathbb{H} \cup \{\infty\} \to X$, a <u>disc bubble</u>.

4) Intermediate bubblings $\Rightarrow$ might need various intermediate rescalings to "catch all bubbles". Moreover, we need to show these bubbles <u>connect up</u>.

This process is a finite process because

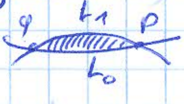
(a) the energy is preserved under all limits

(b) there is an a priori energy estimate:

**Theorem** (a priori energy bound) if $u$ is not constant and $J$-hol,
$E(u) = \int u^* \omega \geq \hbar(X, \omega, J, L) > 0$.

So, each new bubble "drains" $\geq \hbar$ energy. Since energy $(u_n) \leq K$, the process is finite. This bound comes from the fact that low energy $J$-curves satisfy a mean-value inequality, so sufficiently low energy $J$-curves are constant.

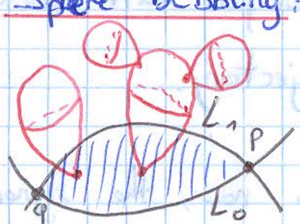
How to compute $\hbar$? Monotonicity, minimal surfaces, ...

**Gromov compactness for Floer trajectories**: say $u_n$ is a sequence of Floer trajectories, e.g. $J$-hol. maps $\mathbb{D}^2 \setminus \{\pm 1\} \to (X, L_0, L_1)$ with finite energy .



3 types of phenomena arise in rescaling / energy blow-up analysis

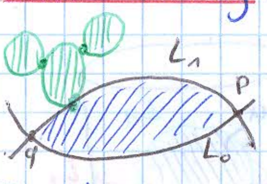
1) Sphere bubbling: in good cases (if these loci are cut out transversely), this happens in "codimension $\geq 2$", so it should not contribute to $\partial \mathcal{M}(p,q)$.





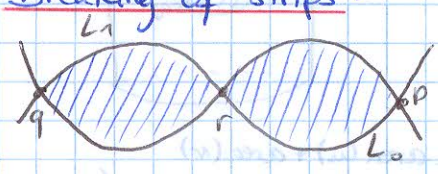
In even nicer cases, this is a priori excluded (if $\pi_2(M) = 0$, or $\langle \omega, \pi_2(M) \rangle = 0$, since $\int u^* \omega > 0$ for non-constant $J$-spheres).

2) Disc bubbling: serious issue: it can (and does) occur "in codimension 1", even when discs are transversely cut out, and this contributes to $\partial \mathcal{M}$.





3) Breaking of strips: "energy escapes to $s = \pm\infty$ on $\mathbb{R} \times [0,1]$ or to $\pm 1$ on $\mathbb{D}^2 \setminus \{\pm 1\}$", eg reparametrizing $u_n(\cdot - s_n, \cdot)$ gives non-equal limit.





This is analogous to Morse theory, where broken flow line



Transversality: this happens in codim 1, so doesn't appear in the compactification of $\frac{\mathcal{M}(p,q,\beta,J)}{\mathbb{R}}$ if $\mathrm{ind}(\beta) = 1$.

We defined $CF^*(L_0, L_1; J) = \Lambda^{|L_0 \cap L_1|}$, with

$$\partial(p) = \sum_{\substack{q,\ \beta \in \pi_2(p,q) \\ \mathrm{ind}(\beta)=1}} \#\left(\mathcal{M}(p,q,\beta,J)/\mathbb{R}\right) \cdot T^{\omega(\beta)} \cdot q$$

Compactness and transversality analysis imply that for generic $J$, this count is finite and well-defined (if no disc / sphere bubbling).

How to prove $\partial^2 = 0$, assuming no bubbling:

Consider $\mathcal{M}(p,q,\beta,J)/\mathbb{R}$ with $\mathrm{ind}(\beta) = 2$. This should be a 1-manifold, which can be compactified to $\overline{\mathcal{M}}(p,q,\beta,J)$ adding in broken trajectories:

$$(*) \quad \coprod_{\substack{r \in L_0 \cap L_1 \\ \beta_1 \# \beta_2 = \beta}} \left(\mathcal{M}(p,r,\beta_1,J)/\mathbb{R}\right) \times \left(\mathcal{M}(r,q,\beta_2,J)/\mathbb{R}\right).$$

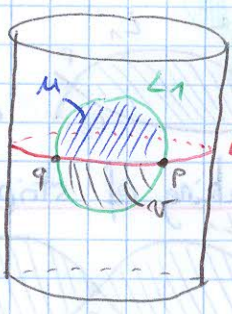
(No other bubbling $\Rightarrow$ no other potential limits $\Rightarrow$ $\overline{\mathcal{M}}$ compact).

**Theorem [gluing]:** The resulting $\overline{\mathcal{M}}$ is a 1-manifold with boundary. So, $(*)$ is a breaking of some smooth trajectory.

$\overline{\mathcal{M}}$ is oriented (we'll talk about that later), so now, the signed number of ends (of any compact 1-manifold), counted with the induced orientation, is always 0. So, for a fixed $q$,

$$0 = \sum_r \sum_{\substack{\beta_1, \beta_2 \\ \beta_1 + \beta_2 = \beta}} T^{\omega(\beta)} \left(\#\mathcal{M}(p,r,\beta_1,J)/\mathbb{R}\right)\left(\#\mathcal{M}(r,q,\beta_2,J)/\mathbb{R}\right)$$

$$= \text{coefficient of } q \text{ in } \partial^2(p).$$

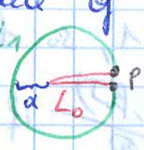



Example when $\partial^2 \neq 0$: $T^*S^1$

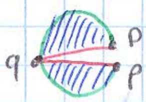
$CF^*(L_0, L_1) = \Lambda p \oplus \Lambda q$

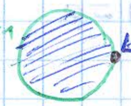
$\partial p = \pm T^{\mathrm{area}(u)} q$

$\partial q = \pm T^{\mathrm{area}(v)} p \Rightarrow \partial^2 p = T^{\mathrm{area}(u) + \mathrm{area}(v)} p \neq 0.$

Look at moduli space of index 2 discs from $p$ to itself. It is an interval: 

. We can find an explicit disk for each $\alpha \in (0,1)$.

For example, in local coordinates, $\dfrac{z^2+\alpha}{1+\alpha z^2}$. There are two endpoints: • $\alpha \to 0$:  $p \to q \to p$ contributes to $\langle \partial^2 p, p \rangle$.

• $\alpha \to 1$:  constant strip with disc bubble attached with boundary on $L_1$.

Suppose no disc / sphere bubbles, so $\partial^2 = 0$. So, we get a cohomology group $HF^*(L_0, L_1; J)$.

Theorems: if no disc / sphere bubbling, $HF^*(L_0, L_1; J)$ is
(a) Independant of $J$; call it $HF^*(L_0, L_1)$.
(b) Hamiltonian isotopy invariant: $HF^*(L_0, L_1) \cong HF^*(\phi_{H_0} L_0, \phi_{H_1} L_1)$
(provided these are transverse.

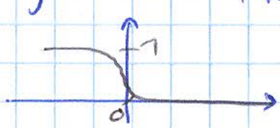
Sketch of proof: in either case (say, we do both at once), we define continuation maps

$$\Phi_{H, J_s, \beta} : CF^*(L_0, L_1; J_0) \longrightarrow CF^*(\phi_H(L_0), L_1; J_1)$$ which are

↳ family of compatible a.c.s, interpolating between $J_0$ and $J_1$.

(i) chain maps for generic choices

(ii) chain isomorphisms: construct $\phi_{-H, J_{-s}} : CF^*(\phi_H L_0, L_1; J_1) \to CF^*(L_0, L_1, J_0)$ and argue that $\phi_{H, J_s} \circ \phi_{-H, J_{-s}} - id = \partial K + K \partial$.

Say, $H : [0,1] \times X \to \mathbb{R}$ generates $\phi_H^t$ flow of $X_H$. Pick a cutoff $\beta : \mathbb{R} \to [0,1)$  and a path $J_s$, $s \in \mathbb{R}$, st $J_s = \begin{cases} J_0 & s \gg 0 \\ J_1 & s \ll 1 \end{cases}$.

Consider, for $p \in L_0 \cap L_1$, $q \in \phi_H L_0 \cap L_1$, finite energy solutions to

$\cancel{\mathbb{R}}$

$$\begin{cases} u : \mathbb{R}_s \times [0,1]_t \to M \\ u(s,i) \in L_i, \quad i \in \{0,1\} \\ \dfrac{\partial u}{\partial s} + J_s\left(\dfrac{\partial u}{\partial t} - \beta(s) X_{H_t}\right) = 0 \\ \lim\limits_{s \to \pm\infty} u(s,t) = \begin{cases} p & \text{if } +\infty \\ \tilde{q} & \text{if } -\infty \end{cases} \end{cases}$$

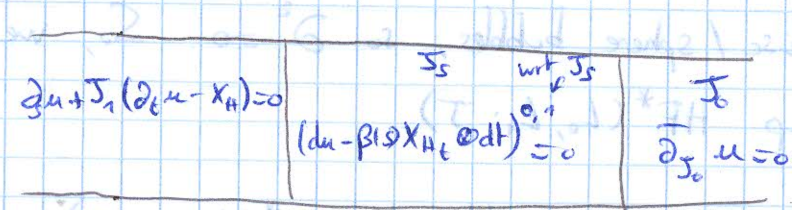
"inhomogeneous CR equation"

"Floer's equation"

time-1 Ham chords of $X_H$ ending at $q$

$\tilde{q} : [0,1] \to X$

$\dot{\tilde{q}} = X_H, \ \tilde{q}(1) = q$

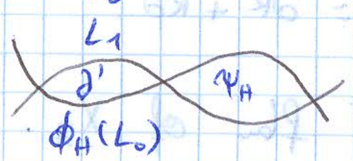


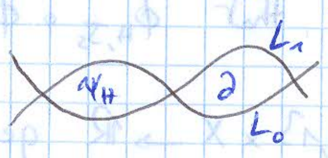

$\uparrow$ p constant path

Rem: by a gauge transformation $\tilde{u}(s,t) := \phi_{H_t}^t u(s,t)$, the solutions of $(*)$ near $-\infty$ are equivalent to solutions $\tilde{u}$ of

$$\begin{cases} \tilde{u}(s,0) \in \phi_H(L_0) \\ \tilde{u}(s,1) \in L_1 \\ \bar\partial_{J_1} \tilde{u} = 0 \\ \lim\limits_{s \to -\infty} \tilde{u}(s,t) = p \end{cases}$$

A count of index 0 solutions (no $\mathbb{R}$ action, are isolated) weighted by energy gives $\Psi_{H, J_s, \beta}$. To see that it's a chain map, compactify and look at codimension 1 boundary:




"$\square$"