08/04/16 : Today: * remarks on stable maps (a) invariance

* Floer's equation vs J-hol strips : (b) Floer's thm: $HF^*(L,L) \overset{\text{sometimes}}{\cong} H^*(L)$

(* Gradings) (c) Oh spectral sequence

A J-hol map $u : \overset{\text{nodal, with marked points}}{\Sigma_\infty} \longrightarrow X$ is called a prestable map : it is stable if its automorphism group is finite, where a automorphism is

$$\begin{tikzcd} \Sigma_\infty \arrow[r, "f"] \arrow[dr, "u"'] & \Sigma_\infty \arrow[d, "u"] \\ & X \end{tikzcd}$$

where f is the underlying automorphism of trees + biholomorphic on components, preserve marked points.

Rem: if $u : \Sigma_\infty \to X$ is constant, then stability of u implies stability of the domain Σ_∞ (equipped with marked points). If u is constant, then u is stable $\Leftrightarrow$ Σ_∞ is stable.

ex:  $\xrightarrow{u} X$ non-constant is stable, because any automorphism $\phi : \mathbb{R} \times [0,1] \circlearrowleft$ is translation, and $u \circ \phi \neq u$.

Rem: in Gromov's compactness, bubbles are only well defined up to overall automorphism $\Rightarrow$ get an element of $\mathcal{M}_{\text{stable}} / \text{Aut of domains}$.
"Aut of domains" has a nearly free action when the maps are stable.

Floer's equation:

It is helpful to have a more flexible way of constructing Floer homology groups, using an auxiliary Hamiltonian.

Define, for $H : [0,1] \times X \to \mathbb{R}$ Hamiltonian, J a.c.s, L_0 and L_1:
$CF^*(L_0, L_1; H, J)$, defined whenever $\phi_H^1(L_0) \pitchfork L_1$.

Let $\mathcal{X}^H_{L_0,L_1} := \{$time-1 chords of X_H from L_0 to L_1
$\gamma : [0,1] \to X$, $\gamma(i) \in L_i$ for $i \in \{0,1\}$, $\dot{\gamma} = X_H\}$.

Define $CF^*(L_0, L_1; H, J) = \Lambda^{|\mathcal{X}^H_{L_0,L_1}|}$.

Differential: for $x^+, x^- \in \mathcal{X}^H_{L_0,L_1}$, $\beta \in \pi_2(x^+, x^-, \ldots)$.

$$\mathcal{M}_H(x^+, x^-, \beta) \begin{cases} u : \mathbb{R} \times [0,1] \longrightarrow X & S := \mathbb{R} \times [0,1] \quad (S, j) \\ u(s, i) \in L_i \\ \lim_{s \to \pm\infty} u(s,t) = x^\pm(t) \\ \partial_s u + J(\partial_t u - X_H) = 0 : \text{Floer's equation} \\ E(u) = \int_S |\partial_t u - X|^2 = \int_S (u^*\omega - d(u^* H\, dt)) < \infty . \end{cases}$$

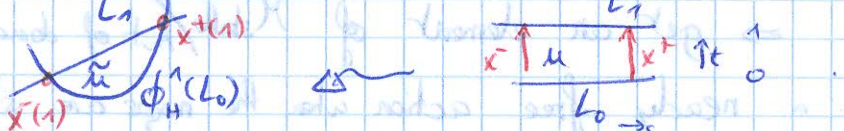
Coordinate free Floer's equation: $(du - X_H \otimes dt)^{0,1} = 0$ ← wrt J, j

where for $f \in \mathrm{Hom}(TS, u^*TX)$, have $(f)^{0,1} = \frac{1}{2}(f + J \circ f \circ j)$.

These are manifolds of index $\mathrm{ind}(\beta)$ for generic J; look at $\mathcal{M}_H(x^+, x^-, \beta)/\mathbb{R}$ when $\mathrm{ind}(\beta) \geq 1$. Then Gromov compactify.

Define $d(x^+) = \sum_{\substack{x^-, \beta \\ \text{st } \mathrm{ind}(\beta) = 1}} T^{E(\beta)} \# \left(\mathcal{M}_H(x^+, x^-, \beta)/\mathbb{R}\right) \cdot x^-$. Gromov compactness and gluing imply $d^2 = 0$ (in absence of bad bubbles).

Note: given $u \in \mathcal{M}_H(x^+, x^-)$, gauge transform $\tilde{u}(s,t) = \phi_H^{1-t} u(s,t)$



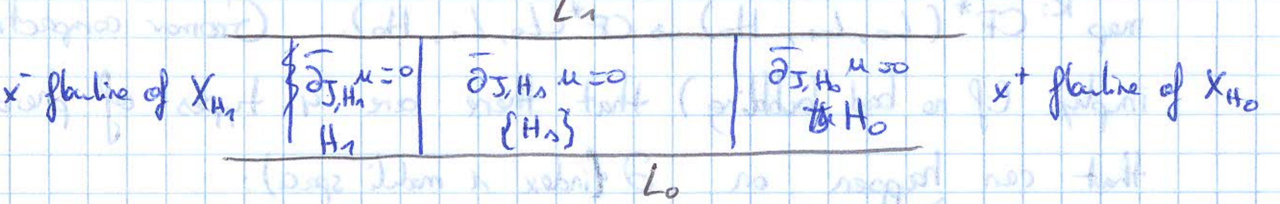
Then, $\tilde{u}$ is a solution to $\bar{\partial}_{\tilde{J}} \tilde{u} = 0$ with boundaries on $\phi_H(L_0)$, L_1, asymptotic to $x^+(1)$, $x^-(1)$, where $\tilde{J} = (\phi_H^{1-t})_* J (\phi_H^{1-t})_*^{-1}$ depends on t (in general, we may have needed t-dependent J anyway for transversality).

$\Rightarrow CF^*(L_0, L_1; H, J) \cong CF^*(\phi_H(L_0), L_1; H, \tilde{J})$ as chain complexes.

As an example of application, we look at continuation maps $CF^*(L_0, L_1; H_0, J) \to CF^*(L_0, L_1, H_1, J)$. If these are quasi-isos, then $HF^*(\phi_H(L_0), L_1) \cong HF^*(L_0, L_1)$.

Define $\bar{\partial}_{J,H} u := (du - X_H \otimes dt)^{0,1} \nwarrow$ for J.

Count index 0 solutions to



Chain map (if no bad bubbling)? ~~The~~ Follows from Gromov compactness / gluing, applied to index 1 moduli spaces. Namely, ∂(index 1 moduli spaces) is

$$\underline{\overline{H_1 \,|\, H_s \,|\, H_0}} \left(\underline{\overline{H_0}} /\mathbb{R} \right) \amalg \left(\underline{\overline{H_1}} /\mathbb{R} \right) \underline{\overline{H_1 \,|\, H_s \,|\, H_0}}$$

$$\Rightarrow \Psi_{H_0,H_1} \circ \partial_{H_0} - \partial_{H_1} \circ \Psi_{H_0,H_1} = 0. \qquad \square$$

$$\Psi_{H_0,H_1}(x^+) = \sum_{\substack{x^-,\beta \\ \mathrm{ind}(\beta)=0}} T^{E(\beta)} \cdot \# \mathcal{M}_{\{H_s\}}(x^+,x^-)\, x^-.$$

<u>Rem:</u> such a map can have negative energy, namely

$E(u) = \int u^*\omega - d(u^* H dt)$

now equals $\int_S |\partial_t u - X_H|^2 - \overbrace{\int_S \partial_s H \, ds \wedge dt}$ → bounded, indep of u ⇒ don't get powers too negative. (*)

$(d^{\text{vert}} u^* H dt + d^{\text{horiz}} u^* H dt = \partial_s H \, ds \wedge dt)$

In general, the continuation maps involve negative powers of T (so, invariance requires Novikov <u>field</u>, not ring).

↳ powers bounded below, ok by (*)

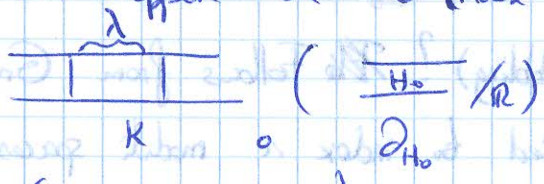
$\Psi_{H_1 H_0} \circ \Psi_{H_0 H_1}$ counts $x^- \underline{\overline{H_0 \,|\, \tilde H_{-s} \,|\, H_1}} \quad \underline{\overline{H_1 \,|\, \tilde H_s \,|\, H_0}}\ x^+$.

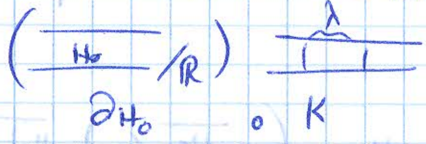
"Homotopy of homotopies": to show $\Psi_{H_1 H_0} \circ \Psi_{H_0 H_1} - \mathrm{id} = \partial K + K \partial$, let $\mathcal{M}_{\lambda,\{H_s\}}$ denote the space of pairs $\{\lambda \in [0,\infty)$, u solution to $x^- \underline{\overline{H_0 | \qquad | H_0}}\ x^+\}$.

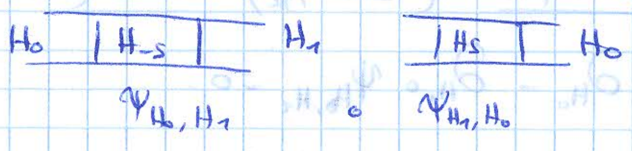
If $\lambda < 1$, then $\underline{\overbrace{| H_0 |}^{\lambda}}$

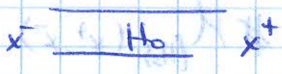
If $\lambda > 2R+1$, then $\underline{\overbrace{|\tilde H_{-s}}^{R} \overbrace{| H_1 |}^{\lambda-2R} \overbrace{\tilde H_s |}^{R}}$.

A count of index 0 elements of $\mathcal{M}_{\lambda,\{H_s\}}$ will define a map $K: CF^*(L_0, L_1; H_0) \to CF^*(L_0, L_1, H_0)$. Gromov compactness + gluing imply (if no bad bubbling) that there are 4 types of phenomena that can happen on ∂(index 1 moduli space):

1)


2)


3) $\lambda \to \infty$:


4) $\lambda \to 0$:


. But: index 0 solutions (not /ℝ).

Claim: the index 0 elements of $\mathcal{M}(x^+, x^-)$ are exactly constant maps ($\Rightarrow$ count gives Id).

So, mod signs, get $K \circ \partial_{H_0} - \partial_{H_0} \circ K + Id - \Psi_{H_1,H_0} \circ \Psi_{H_0 H_1} = 0$. The reverse composition is identical computations, so we get that $\Psi_{H_0 H_1}$ is a homology iso.

Rem: Invariance $\Rightarrow$ define $HF^*(L, L) := HF^*(L, L, H, J)$ for H with $\phi_H(L) \pitchfork L$. Invariance for J is similar.

ex: $X = T^*Q$, $\omega_{can} = d\lambda_{can} = dp \wedge dq$, $L \subseteq X$ 0-section.
$L \subseteq X$ is exact, meaning $\lambda_{can}|_L = df$ (in fact, $\lambda_{can}|_L = 0$).
In particular, Stokes $\Rightarrow \pi_2(T^*Q) = \pi_2(T^*Q, L) = 0$.
So, $HF^*(L, L)$ is well-defined, as long as Gromov compactness holds. (Gromov compactness requires all strips to be mapped to $C \subseteq T^*Q$ compact, but in principle, strips could escape to ∞ in target).

Claim: have an a priori C^0 estimate on Floer curves

$u: \mathbb{R} \times [0,1] \to (T^*Q, L)$.

It comes from "maximum principle", "monotonicity" (T^*Q is convex at ∞, Liouville, ...).

Assuming the claim, we have $HF^*(L,L)$

Theorem [Floer]: $HF^*(L,L) \cong H^*(L)$

(more generally true for any $L \subseteq X$, provided $\pi_2(X) = 0 = \pi_2(X,L)$).

Proof: by invariance, compare $CF^*(L,L;H,J)$ with $C^*(L) = CM^*(f,g)$ for specific nice H, J. Pick g metric on L, $f: L \to \mathbb{R}$ Morse function with (f,g) "Morse-Smale".

Note: g on $L = Q$ induces a splitting $TT^*Q \cong T_{vert}T^*Q \oplus T_{horiz}T^*Q$, and induces a J on TT^*Q.

On $T(T^*Q)|_Q \underset{g}{\cong} T^*Q \oplus TQ$;

J on $\uparrow$ is the natural pairing $\uparrow$ induced by g:

$$J(\underset{T_x^*Q}{\underset{\cap}{\phi}}) = g(\underset{T_xQ}{\underset{\cap}{\phi}}, -).$$

Note: $f: L \to \mathbb{R}$ induces a Hamiltonian H:

$$\begin{tikzcd} T^*Q \arrow[r, "H"] \arrow[d, "\pi"'] & \mathbb{R} \\ Q = L \arrow[ur, "f"'] & \end{tikzcd}$$

(maybe cutoff H near ∞).

Theorem [Floer]: if f is C^2 small, there is a bijection

$$\left\{\begin{matrix} \gamma: \mathbb{R} \to Q, \ \dot\gamma(s) = \nabla f_{\gamma(s)} \\ \text{flowlines for } (f,g) \end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix} \text{solutions to Floer's equation} \\ \partial_s u + J(\partial_t u - X_H) = 0, \\ \text{this } H, J, \text{ asymptotics } x^\pm \end{matrix}\right\}$$

with asymptotics $x^\pm$

Note: in coordinates, $H(q,p) = f(q)$

$\Rightarrow dH = f'(q)\,dq$

$\Rightarrow X_H = f'(q)\,\partial_p$

(get $JX_H|_Q = \nabla f$ in the TQ piece of $TT^*Q|_Q$).

Note: $X_H = 0$ at critical points of $f: Q \to \mathbb{R}$, so

$$\left\{\begin{array}{l}\text{constant trajectories}\\ \text{of } X_H \text{ from } L \text{ to } L\end{array}\right\} \longleftrightarrow \mathrm{crit}(f).$$

Further: if $\gamma: \mathbb{R}_s \to Q$ is st $\dot\gamma(s) = \nabla f_{\gamma(s)}$, $u(s,t) := \gamma(s)$ satisfies

$$\underbrace{\partial_s u}_{\dot\gamma(s)} + \underbrace{J(\underbrace{\partial_t u}_{=0} - X)}_{= -JX = -\nabla f} = 0$$

(+ argue they are all constant) □

In general for $\pi_2(X)$, $\pi_2(X, L) \neq 0$, there may be discs/spheres classes, and in particular more classes of strips, by "connect sums" of homotopy classes:

$v \in \pi_2(X, L)$

If d happens to be well-defined and $d^2 = 0$, one can look at the energy filtration of terms contributing to d.

Because

(a) Near any $L \subseteq X$, $\exists$ Weinstein nbhd $U \cong T^*L$ of L

(b) "low energy strips/discs" must stay inside U (monotonicity lemma),

the low energy part of d coincides (for nice H, J) with the the Morse differential, by Floer's argument. ↳ in $CF^*(L, L, H, J)$

$\Rightarrow \exists$ spectral sequence $H^*(L) \Rightarrow HF^*(L, L)$ [Oh spectral sequence]

More generally, if L_0, L_1 have clean intersection so that $L_0 \cap L_1 = N^k \subseteq X$, Pozniak constructed a spectral sequence (under hypotheses of definedness)

$H^*(N) \Rightarrow HF^*(L_0, L_1)$, coming from a reduction to a local model in low energy: $Q \hookrightarrow T^*Q$ zero section

$$\begin{array}{c} T^*Q \\ \cup \\ L_N \text{ conormal to } N \\ \| \\ \nu^*(N) := \{(q,p) \mid q \in N,\ p \text{ annihilates } TN\}. \end{array}$$