11/04/16

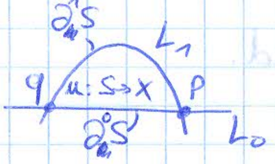
<u>Today</u>: 1) gradings
2) product structures
3) signs

<u>Recall</u>: $L_0, L_1 \subseteq (X,\omega)$, J a.c.s., $H: [0,1]\times X \to \mathbb{R}$ (possibly 0).
We defined $CF^*(L_0,L_1; H, J) := \Lambda^{|\phi_H L_0 \cap L_1|}$ ← think of this as time-1 chords $L_0 \to L_1$ of X_H,
with $\delta p := \sum_{\substack{q,\ \beta\in\pi_2(p,q)\\ \text{ind}\,\beta = 1}} T^{E(\beta)} \#\left(\mathcal{M}(p,q)/\mathbb{R}\right)\cdot q$.

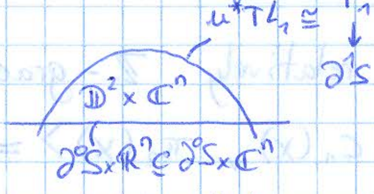
In nice cases, δ is well-defined, $\delta^2 = 0$, $HF^*(L_0,L_1)$ does not depend on H and J.

Gradings: (assume $H=0$ for simplicity)

For $[u]\in\pi_2(p,q) = \pi_2(X; L_0, L_1, p, q)$, we defined $\text{ind}([u])$ as follows:



Simultaneously trivialize u^*TX and $(u|_{\partial^0 S})^*TL_0$, to get:

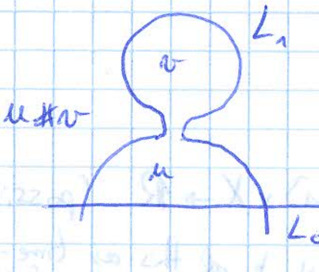


So get $\partial^1 S \to \Lambda(n)$ Grassmannian of linear Lagrangians $\subseteq \mathbb{C}^n$,
e.g. a path γ_t in $\Lambda(n)$ which is transverse to $\mathbb{R}^n \subseteq \mathbb{C}^n$ at 0 and 1
↳ define $\text{ind}([u]) := \gamma_t \cdot \underbrace{\Lambda_1(n)}_{:= \{L\subseteq\mathbb{C}^n \text{ linear Lagr} \mid L\cap\mathbb{R}^n \neq \{0\}\}}$.

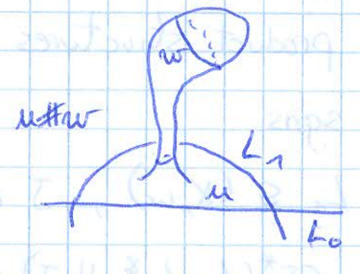
<u>Want</u>: assign an absolute $\mathbb{Z}$-grading to a $p\in L_0\cap L_1$, called $\deg(p)$, so that $\deg(q) - \deg(p) = \text{ind}([u])$ for any $u\in\pi_2(p,q)$.
Of course, this is impossible if there are $\pi_2(p,q)$ classes whose indices differ: $[u],[v]\in\pi_2(p,q)$ s.t. $\text{ind}([u]) \neq \text{ind}([v])$.

<u>Sources of ambiguity</u>: in $\text{ind}(\cdot)$:
Given $[u]\in\pi_2(p,q)$ and $v: (\mathbb{D}^2, S^1)\to(X, L_i)$ $\quad v\in\pi_2(X,L_i)$,
or $w: S^2 \to X$ $\quad w\in\pi_2(X)$

Connect sum give new elements of $\pi_2(p,q)$:



or



For disc: $\operatorname{ind}([u\#v]) = \operatorname{ind}(u) + \mu(v)$ (Maslov index)

For sphere: $\operatorname{ind}([u\#w]) = \operatorname{ind}(u) + 2\langle c_1(TX), w_*[S^2]\rangle$

So in general, we can only even associate relative gradings in $\mathbb{Z}/N\mathbb{Z}$, where $N\mathbb{Z}$ is the subgroup generated by these ambiguity terms $\mu(v)$ and $2\langle c_1(TX), w_*[S^2]\rangle$.

<u>Exercise</u>: if L_0, L_1 oriented, then $\overset{\text{disc class in } \pi_2(X,L_i)}{\mu(v)} \in 2\mathbb{Z}$, so $2 \mid N$ and $CF^*(L_0,L_1)$ is relatively $\mathbb{Z}_2$-graded.

We can relatively $\mathbb{Z}$-grade if

(a) $\langle 2c_1(X), \pi_2(X)\rangle = 0$

(b) $\mu(v) = 0$. If (a), we can think of it as $\tilde{\mu}: H_1(L_i) \to \mathbb{Z}$, via $\mu(\rho \in \pi_2(X,L_i)) = \tilde{\mu}(\partial \rho)$. This is well-defined if (a), since two ρ's with same $\partial\rho$ differ by a sphere. We want $\tilde{\mu}$ to vanish.

<u>Absolute gradings</u> [Kontsevich, Seidel]

Fixing extra data on X, L_i, we can associate absolute $\mathbb{Z}$-gradings to $CF^*(L_0,L_1)$.

<u>Idea</u>: $\exists$ choices of paths between $\Lambda_0, \Lambda_1 \in \Lambda(n)$, but not between $\Lambda_0^\#, \Lambda_1^\# \in \tilde{\Lambda}(n)$ the universal cover.

<u>Recall</u>: Lagr. Grassmannian $\Lambda(n) \cong U(n)/O(n)$, $H^1(\Lambda(n);\mathbb{Z}) = \mathbb{Z}\langle \mu\rangle$ and $\pi_1(\Lambda(n)) \cong \mathbb{Z}$, with $\det^2: U(n) \to S^1$ a π_1-iso which classifies μ.

We have $\widetilde{\Lambda}(n) \to \Lambda(n)$ the universal cover.

Global case. X symplectic manifold, $\begin{matrix}\mathcal{L}\\ \downarrow \\ X\end{matrix}$ bundle of linear Lagrangian subspaces in $(T_x X, \omega)$, so $\mathcal{L}_x \cong \Lambda(n)$, non-canonically.

Proposition: the following are equivalent:

(i) $\exists$ global $\mu_{glob} \in H^1(\mathcal{L}; \mathbb{Z})$ restricting to the Maslov class generator $\mu_x \in H^1(\mathcal{L}_x; \mathbb{Z})$ $\forall x$.

(ii) $\exists$ cover $\widetilde{\mathcal{L}} \to \mathcal{L}$ whose restriction to each $\mathcal{L}_x$ is a universal cover

$$\begin{array}{ccc} \widetilde{\mathcal{L}}_x & \longrightarrow & \mathcal{L}_x \\ \wr\wr & & \wr\wr \\ \widetilde{\Lambda(n)} & \longrightarrow & \Lambda(n) \end{array}$$

(iii) $\exists$ a trivialisation of the square of the determinant bundle
$$(\Lambda^n_{\mathbb{C}} TX)^{\otimes 2} \cong K_x^{-\otimes 2} \cong \mathbb{C}. \qquad (\text{holds if } 2c_1(X) = 0)$$

In fact, choices of (i), (ii), (iii) are canonically identified.

Rem: in fact, choices of (i), (ii), (iii) are each affine spaces over $H^1(X; \mathbb{Z}) \cong [X, S^1]$ (exercise).

Let us fix such a choice. Now, to each Lagrangian $L \subseteq X$, there is a canonical section $s_L : L \to \mathcal{L}|_L$, associating to each point of the Lagrangian its tangent space.

$$\begin{tikzcd} & \mathcal{L} \arrow[d] \\ L \arrow[ur, "s_L"] \arrow[r, hook] & X \end{tikzcd}$$

Given this, the Maslov class of L is $\mu_L := s_L^* \mu_{glob} \in H^1(L; \mathbb{Z})$.

Definition: a $\underline{\mathbb{Z}\text{-grading}}$ of L is a lift

$$\begin{tikzcd} & \widetilde{\mathcal{L}}|_L \arrow[d] \\ L \arrow[ur, dashed, "\tilde{s}_L"] \arrow[r, "s_L"'] & \mathcal{L}|_L \end{tikzcd}$$

The obstruction to $\tilde{s}_L$ existing is $\mu_L \in H^1(L; \mathbb{Z})$.

Hence, gradings exist iff $\mu_L = 0$ (L is Maslov zero).

Rem: any $L \cong S^n$ $(n>1)$ or with $H^1(L) \cong 0$ is automatically Maslov zero.

Rem: there exists a notion of an absolute $\mathbb{Z}/N\mathbb{Z}$-grading structure: replace $\tilde{\mathcal{L}}$ with a N-fold fibrewise cover $\tilde{\mathcal{L}}^N$ .... Instead of trivializing $K_X^{-\otimes 2}$, choose a N^{th} root of it.

Given $\mathbb{Z}$-grading structures on L_0, L_1, how to define $\deg(\overset{L_0\cap L_1}{\overset{\in}{p}})$?

Have $\underset{\Lambda_0(t)}{\overset{\shortparallel}{T_p\tilde{L}_0}}, \underset{\Lambda_1(t)}{\overset{\shortparallel}{T_p\tilde{L}_1}} \in \tilde{\mathcal{L}}_p \cong \tilde{\Lambda}(n)$

with $\Lambda_0(0) = \Lambda_1(0) =$ basepoint.

Projecting to $\Lambda(n)$ induces paths $\lambda_0(t), \lambda_1(t)$.

Definition: $\underline{\deg(p)} := n/2 - \mu(\lambda_0, \lambda_1)$

to make it agree with Morse cohomology. number of times $\lambda_0 \not\pitchfork \lambda_1$, counted with sign.

<u>Easier to compute point of view</u>: suppose $c_1(X) = 0$ (for instance, X Kähler).
Take some complex volume form Ω_X (nowhere vanishing $(n,0)$ form $\Leftrightarrow$ non-zero section of $\Lambda^n_{\mathbb{C}} TX$). This gives a trivialization of $K_X^{-\otimes 1}$, so also a trivialization of $K_X^{-\otimes 2}$.
(rem: $c_1 = 0 \Rightarrow$ topological section, maybe not holomorphic, so Ω_X maybe not holomorphic. Let's take it holomorphic).

$\hookrightarrow$ Get a <u>classical phase function</u> $\alpha_X : \mathcal{L} \longrightarrow S^1$: given $x \in X$, $\Lambda \in \mathcal{L}_x$, pick a basis $e_1, \dots, e_n$ of Λ, and define

$$\alpha_X := \frac{\Omega_X(e_1 \wedge \dots \wedge e_n)^2}{|\Omega_X(e_1 \wedge \dots \wedge e_n)|^2}.$$

Rem: independent of basis.

Note that $(\alpha_X)^*(\text{gen. of } H^1(S^1)) = \mu_{glob} \in H^1(\mathcal{L}; \mathbb{Z})$

A grading of $L \Leftrightarrow$ a choice of lift of

$$\begin{tikzcd}
 & & \mathbb{R} \arrow[d, "\exp(2i\pi -)"] \\
L \arrow[r, "s_L"] \arrow[urr, dashed, "\tilde{\alpha}_L"] \arrow[rr, bend right, "\alpha_L"'] & \mathcal{L}|_L \arrow[r, "\alpha_X"] & S^1
\end{tikzcd}$$

α_L classifies $\mu_L \in H^1(L; \mathbb{Z})$

Call $\tilde{L} := (L, \tilde{\alpha}_L)$ a graded Lagrangian.

* Can shift: $\tilde{L}[1] := (L, \tilde{\alpha}_L + 1)$
* Given $p \in \tilde{L}_0 \cap \tilde{L}_1$, choose a path Λ_t from T_pL_0 to T_pL_1 in $\mathcal{L}_p$ which is crossingless ("canonical short path"). Lift $\alpha_X(\Lambda_t)$ to $\tilde{\alpha}_t \in \mathbb{R}$. Then, $\deg(p) = (\tilde{\alpha}_{L_1}(p) - \tilde{\alpha}_1) - (\tilde{\alpha}_{L_0}(p) - \tilde{\alpha}_0)$.

Rem: the "$-\tilde{\alpha}_1$" gets us a integer.

Rem: had to make a choice for X (triv. of $\det^2$ bundle), and choices for Lagrangians (lift them). Eventually, objects of the $\mathbb{Z}$-graded Fukaya category wrt a chosen trivialization $K_X^{-\otimes 2} \cong \mathbb{C}$ are pairs $(L, \tilde{\alpha}_L, \dots)$.

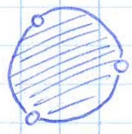
Product structures

[Donaldson]: can define a map (in nice cases)

$$[\mu^2] : HF^*(L_1, L_2) \otimes HF^*(L_0, L_1) \longrightarrow HF^*(L_0, L_2).$$

Consider the "open pair of points" $= \mathbb{D}^2 \setminus \{e^{2i\pi k/3}\}$,

and equip it with "strip-like ends".

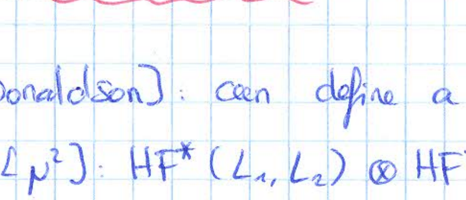


Definition: a positive/negative strip-like end in Σ, at $z_0 \in \partial\Sigma$ is a hol. map

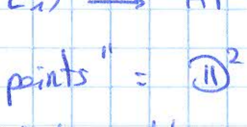
pos: $\varepsilon^+ : [0, +\infty) \times [0,1] \to \Sigma$

neg: $\varepsilon^- : (-\infty, 0] \times [0,1] \to \Sigma$

biholomorphic onto its image, and asymptotic to z_0:



Fix two positive strip-like ends at z_0 and z_1, and one negative at z_2.

This gives us a structure 

. We can study J-hol curve equations on $\Sigma \to X$, which restrict via $(\varepsilon_i^\pm)^*$ to the usual

(34)

$\bar{\partial}$ equation $\partial_s u + J \partial_t u = 0$

Given $x_1 \in CF^*(L_0, L_1, J_0)$, $x_2 \in CF^*(L_1, L_2; J_1)$ and $x_{out} \in CF^*(L_0, L_2; J_2)$, get $\mathcal{M}(x_{out}; x_1, x_2)$ (make some choices of ends, J restricting to J_0, J_1, J_2 on various ends, etc). A count of index 0 solutions will give matrix coefficients for the map $[\mu^2]$ on chain level.