$\bar\partial$ equation $\partial_s u + J \partial_t u = 0$

Given $x_1 \in CF^*(L_0, L_1, J_0)$, $x_2 \in CF^*(L_1, L_2; J_1)$ and $x_{out} \in CF^*(L_0, L_2; J_2)$, get $\mathcal{M}(x_{out}; x_1, x_2)$ (make some choices of ends, J restricting to J_0, J_1, J_2 on various ends, etc). A count of index 0 solutions will give matrix coefficients for the map $[\mu^2]$ on chain level.

13/04/16 Before continuing talking about product structures, let's talk about signs.

Orientations: [Floer-Hofer (Ham. Floer homology), deSilva + FOOO (Lagr. setting), Seidel, Wehrheim-Woodward "Orientations...", Abouzaid's monograph]

<u>Want</u>: to associate a signed count $\#(\mathcal{M}(p,q)/\mathbb{R})$ in $\partial: CF^*(L_0, L_1)^{\circlearrowleft}$.
More generally, we want an orientation of $\mathcal{M}(p,q)/\mathbb{R}$ or by trivializing the $\mathbb{R}$-action, orient $\mathcal{M}(p,q)$.

<u>Meta-statement</u>: (not the most general)$^{(*)}$: $\mathcal{M}(p,q)$ is canonically oriented, relative to certain orientations of 1-dim vector spaces associated to p, q (the orientation lines o_p, o_q), after fixing spin structures on L_0 and L_1.

(*) $L_0, L_1 \subseteq X$, $p, q \in L_0 \pitchfork L_1$, say L_0, L_1 oriented.

<u>Rem</u>: $\mathrm{spin}(n) \xrightarrow{2:1} SO(n)$, is the universal cover if $n \geq 3$.

<u>Rem</u>: more generally, could ask L_i to be relatively Pin, rel $b \in H^2(X; \mathbb{Z}_2)$. Spin corresponds to $b = 0$ and L_i oriented as well, condition $b|_L = w_2(L)$.

So, Floer cohomology (rel b) with signs & grading has objects (L, P spin structure rel b, $\tilde\alpha_L$ grading, ...).

<u>Notation</u>: V finite dimensional vector space $\rightsquigarrow \det(V) = \overset{\lambda(V)}{\overset{\shortparallel}{\Lambda^{top} V}}$, $\mathbb{Z}/2$-graded 1-dimensional vector space, grading is $\dim(V)$.

For M a manifold, get $\lambda(TM)$ the determinant line bundle.
An orientation of $M \longleftrightarrow$ trivialization of $\lambda(TM)$.

For $D : X \to Y$ a Fredholm operator, get $\det(D) = \lambda(\operatorname{coker} D)^* \otimes \lambda(\ker D)$, so get a natural line bundle $\underline{\det} \to \operatorname{Fred}(X,Y)$.

More specific to our setting: $\begin{array}{c}\mathcal{E}\\ \pi\downarrow\uparrow s\\ \mathcal{B}\end{array}$ Banach bundle, with s a section whose linearization is Fredholm.

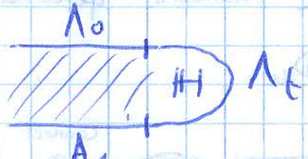
So, get $\begin{array}{c}\underline{\det}\\ \downarrow\\ \mathcal{B}\end{array}$ and $\begin{array}{c}\underline{\det}\\ \downarrow\\ \mathcal{M} = s^{-1}(0)\end{array}$, $\underline{\det}_u \cong \det(D^{vert}_{s_u})$

<u>Note</u>: if $\mathcal{M} = s^{-1}(0)$ is transversally cut out, then $\underline{\det}_u \cong \lambda(\ker D s^{vert}_u) = \lambda(T_u \mathcal{M})$
So, an orientation/trivialization of $\begin{array}{c}\underline{\det}\\ \downarrow\\ \mathcal{M}\end{array}$ is the same as an orientation of $\mathcal{M}$.

The orientation line of $p \in L_0 \cap L_1$ (or more generally, $L_0 \xrightarrow{\gamma} L_1$ time 1 chord of X_H): the idea is, given p, construct a linearized Cauchy-Riemann operator D_p associated to p, and take $o_p := \det(D_p^{local})$.

$p \rightsquigarrow$ a bundle pair (E, F) $\downarrow$ $(\mathbb{D}^2 \setminus \{+1\}, \partial(\mathbb{D}^2 \setminus \{+1\})) \cong (\mathbb{H}, \mathbb{R})$. Fix a negative strip-like end around $+1$ ($\cong +\infty$) :

E is the trivial bundle $\mathring{\mathbb{D}}^2 \times T_p X \cong \mathring{\mathbb{D}}^2 \times \mathbb{C}^n$, and F is a path of Lagrangian subspaces between $\Lambda_0 := T_p L_0$ and $\Lambda_1 := T_p L_1$, constant near ∞:



<u>Rem</u>: if the L_i's come equipped with a grading structure, e.g. lifts $\begin{array}{ccc} & & \widetilde{\mathcal{L}}|_L \\ & \overset{\tilde{\alpha}_L}{\nearrow} & \downarrow \\ L_i & \longrightarrow & \mathcal{L}|_L \end{array}$, then Λ_t is uniquely determined up to homotopy rel endpoints.

Choosing J on E, we get a local Cauchy-Riemann operator $\bar\partial_{(E,F)}$, acting on sections which are exponentially asymptotic near ∞ to $\Lambda_0 \cap \Lambda_1$, i.e.

$$\bar\partial : W^{k,p}(\mathbb{H}, E, F, \underset{``(p,0)" \in T_pX}{0 = \Lambda_0 \cap \Lambda_1}) \to W^{k-1,p}(\mathbb{H}, \Omega^{0,1} E \otimes E).$$

Linearize (at 0) $\bar\partial$, get D_p a Fredholm operator.

Definition: $o_p := \det(D_p)$.

Claims: 1) Independent (up to canonical isomorphism) of choices of J, etc.

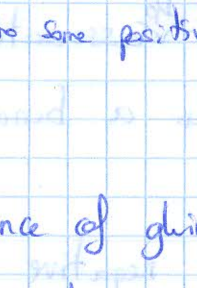
2) An elaboration of the index calculation (showing $\operatorname{ind}(D_p) = \#$ times where $\Lambda_t \not\pitchfork \Lambda_1$ with signs) shows that

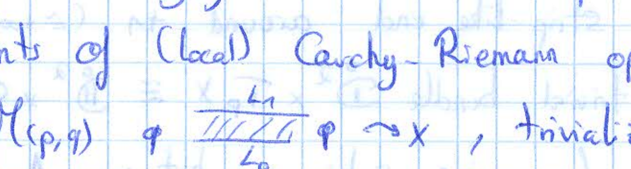
$$\det(D_p) \cong \bigotimes_{\substack{t_i \text{ times that} \\ \Lambda_t \not\pitchfork \Lambda_1,\ \text{ordered}}} V_{t_i}, \quad \text{where } V_{t_i} = \big(\underset{\substack{\uparrow \\ \text{for generic } \Lambda_t,\ \text{this is 1-dim.}}}{\Lambda_t \cap \Lambda_1}\big)^{\otimes \operatorname{sign} \Lambda_t \cap \Lambda_1}$$

Theorem: Let L_0, L_1 oriented, ~~say~~ equipped with fixed Spin structures. Then, there is a canonical isomorphism (up to some positive rescaling)

$$\lambda(T\mathcal{M}(p,q)) \cong o_q \otimes o_p^\vee.$$

How does this work, roughly? It's a consequence of gluing theorems for determinants of (local) Cauchy-Riemann operators:

Given $u \in \mathcal{M}(p,q)$  $\to X$, trivialize u^*TX to obtain a local Cauchy-Riemann operator $\bar\partial_u$:  , and linearize D_u.

Can glue:  , because it was constant near the strip-like end. Finally, homotope $\to$ 

Gluing theorem: $\det(D_p) \otimes \det(D_u) \xrightarrow{\sim} \det(D_p \#_R D_u)$ R-large, gluing together elements of ker / coker using cutoffs $\underset{\text{homotope}}{\cong} \det(D_q)$.

(*) Need to check: independance of

(a) choice of trivialization of $u^* TX$

(b) choice of homotopy

(c) choice of path γ

} Some subset of these requires fixing Spin structures.

(*) Choose one u_0 in each connected component, connect up to an arbitrary $u \in \mathcal{M}(p,q)$ by a path $\gamma : u_0 \rightsquigarrow u$, and use the induced isomorphism from u_0.

Point: a Spin structure on L determines a distinguished class of (stable) trivializations of TL on 1-skeleton which extends to the 2-skeleton.

[Compare: Theorem [de Silva, Fooo] if L is oriented, then a Spin structure on L induces an orientation of $\mathcal{M}(X, L; J) = \{u : (D, S^1) \to (X, L) \mid \bar{\partial}_J u = 0\}$.

Floer theory with signs:

There are 2 options:

(a) "Coherent orientations" [Floer - Hofer], requires some choices of compatible orientations of each o_p.

(b) "Canonical orientations" [Seidel]

We'll do (b), and then see how to get (a) from it.

Let $CF^*(L_0, L_1; H, J) := \bigoplus_{x \in L_0 \cap L_1} |o_x|_\Lambda \leftarrow$ ground field.

Given V a real 1-dim vector space, $|V|_{\mathbb{k}}$ is the free $\mathbb{k}$-module generated by orientations of V, modulo the sum of opposite orientations vanishes.

Note: $|V|_{\mathbb{Z}/2} \cong \mathbb{Z}/2$ canonically.

Given $u \in \mathcal{M}(p,q)/\mathbb{R}$ rigid, $\lambda(T_u \mathcal{M}(p,q)/\mathbb{R}) \cong \underline{\mathbb{R}}$. (trivial bundle over a point)

After trivializing the $\mathbb{R}$-action, get

$$\mathbb{R} \cong \lambda(T_u \mathcal{M}(p,q)) \cong o_q \otimes o_p^\vee ,$$

So we get isomorphism $\mu_u : o_p \to o_q$, which induces

$|\mu_u|_{\mathbb{K}} : |o_p|_{\mathbb{K}} \to |o_q|_{\mathbb{K}}$.

in (a), we say that $\mathrm{sgn}(u) = +1$ if μ_u orientation preserving, -1 otherwise.

So, for $[x] \in |o_p|_\Lambda$, define

$$\partial([x]) = \sum_q \sum_{\substack{\beta \\ \mathrm{ind}(\beta)=1}} \sum_{u \in \mathcal{M}(p,q)/\mathbb{R}} T^{\omega(u)} \cdot \mu_u([x]) \cdot (-1)^{|x|}$$

Why is $\partial^2 = 0$? When $\mathrm{ind}(\beta) = 2$, $\partial(\mathcal{M}(p,q)/\mathbb{R}) = \coprod \mathcal{M}(p,s)/\mathbb{R} \times \mathcal{M}(s,q)/\mathbb{R}$

We have

$$\begin{tikzcd}
\lambda(T\partial \mathcal{M}(p,q)/\mathbb{R}) \arrow[r, "\cong"] \arrow[d] \arrow[dr, phantom, "\circlearrowright \text{ (analyse signs)}"] & \lambda(T\mathcal{M}(p,s)/\mathbb{R}) \otimes \lambda(T\mathcal{M}(s,q)/\mathbb{R}) \arrow[d] \\
o_q \otimes o_p^\vee \arrow[r, "\cong"] & o_q \otimes o_s^\vee \otimes o_s \otimes o_p
\end{tikzcd}$$

This gets even more messy when we look at the product structure...

"And they never mentioned signs again".