Given $u \in \mathcal{M}(p,q)/\mathbb{R}$ rigid, $\lambda(T_u \mathcal{M}(p,q)/\mathbb{R}) \cong \underline{\mathbb{R}}$. (trivial bundle over a point)

After trivializing the $\mathbb{R}$-action, get

$$\mathbb{R} \cong \lambda(T_u \mathcal{M}(p,q)) \cong o_q \otimes o_p^\vee ,$$

so we get isomorphism $\mu_u : o_p \to o_q$, which induces

$|\mu_u|_{\mathbb{K}} : |o_p|_{\mathbb{K}} \to |o_q|_{\mathbb{K}}$.

↖ in (a), we say that $\mathrm{sgn}(u) = +1$ if μ_u orientation preserving, -1 otherwise.

So, for $[x] \in |o_p|_\Lambda$, define

$$\partial([x]) = \sum_q \sum_{\substack{\beta \\ \mathrm{ind}(\beta)=1}} \sum_{u \in \mathcal{M}(p,q)/\mathbb{R}} T^{\omega(u)} \cdot \mu_u([x]) \cdot (-1)^{|x|}$$

Why is $\partial^2 = 0$? When $\mathrm{ind}(\beta) = 2$, $\partial(\mathcal{M}(p,q)/\mathbb{R}) = \coprod \mathcal{M}(p,s)/\mathbb{R} \times \mathcal{M}(s,q)/\mathbb{R}$

We have

$$\begin{tikzcd}
\lambda(T\partial \mathcal{M}(p,q)/\mathbb{R}) \arrow[r, "\cong"] \arrow[d] \arrow[dr, phantom, "\circlearrowright \text{ (analyse signs)}"] & \lambda(T\mathcal{M}(p,s)/\mathbb{R}) \otimes \lambda(T\mathcal{M}(s,q)/\mathbb{R}) \arrow[d] \\
o_q \otimes o_p^\vee \arrow[r, "\cong"] & o_q \otimes o_s^\vee \otimes o_s \otimes o_p
\end{tikzcd}$$

This gets even more messy when we look at the product structure...

"And they never mentioned signs again".

18/04/16 Actually, let's talk about orientations again: how does the (S)Pin structure factor in to various isos/ signed counts?

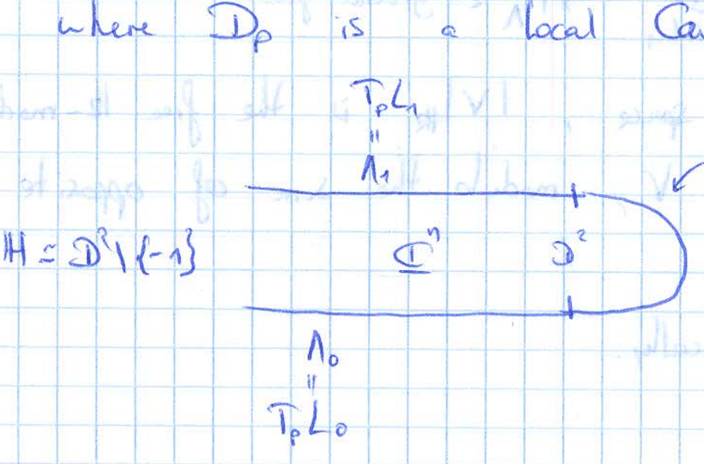
$p \in L_0 \cap L_1 \rightsquigarrow o_p$ orientation line (real $\mathbb{Z}/2\mathbb{Z}$-graded 1D vector space)

$$" \det(D_p)$$

where D_p is a local Cauchy-Riemann operator

$F \subseteq \underline{\mathbb{C}}^n|_{\partial \mathbb{D}^2}$ Lagr. subbundle

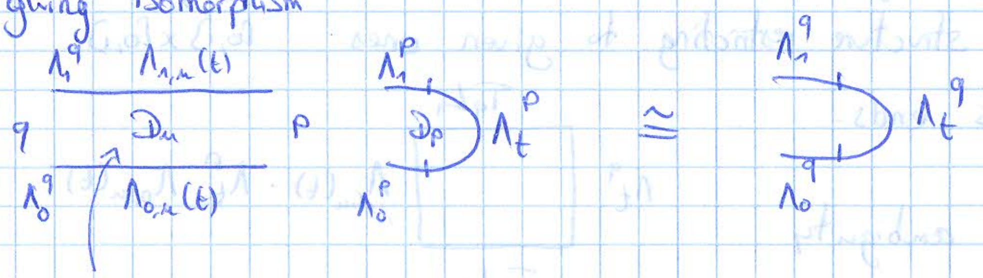
Λ_t path in $\Lambda(n)$.



← choose Λ_t, unique up to homotopy

if L_0, L_1 graded, so Λ_0, Λ_1 lift to $\tilde{\Lambda}(n)$

Given p, q, $u \in \mathcal{M}(p,q)$, we wanted to say there is a gluing isomorphism

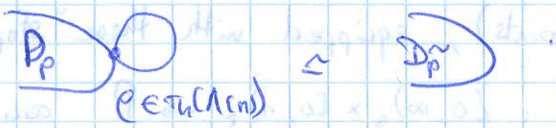


local CR operator associated to u on $u^*TX \simeq \mathbb{C}^n$
$\downarrow$
$\mathbb{R} \times [0,1]$

inducing a map $\det(D_u) \otimes \det(D_p) \longrightarrow \det(D_q)$.

Such an iso depends on a choice of homotopy of paths in $\Lambda(n)$ rel endpoints between Λ_t^q and $\Lambda_{1,u}(t)\, \Lambda_t^p \cdot \Lambda_{0,u}(t)$.

Rem: they are always homotopic if L_0, L_1 are graded. Otherwise, factor in difference by an element of π_1, via

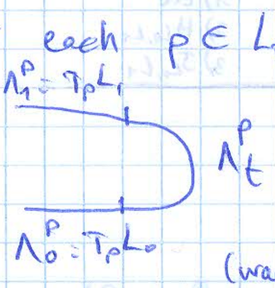


Actually, there is a non-trivial set of choices of homotopy, because

$$\pi_1(\underset{\Omega^2 \Lambda(n)}{\underbrace{P_{a,b}\Lambda(n)}}) = \pi_2(\Lambda(n)) = \mathbb{Z}/2 \quad (n \text{ suff. large} \rightsquigarrow \text{stable range}).$$

The two choices lead to two different isos which differ by a sign.

Solution: equip each L_i with a Spin structure (or rel. Pin structure). For each $p \in L_0 \cap L_1$, make an extra choice: equip each Λ_t^p



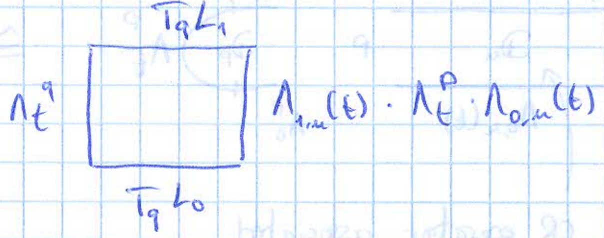
with a Spin structure restricting to given Spin structures already determined at endpoints

(want a Spin structure on $\overset{F}{\underset{[0,1]}{\downarrow}}$ restricting to given one on endpoints).

Wanted: choose a homotopy $\Lambda_t^q \simeq \Lambda_{1,u}(t) \cdot \Lambda_t^p \cdot \Lambda_{0,u}(t)$. When viewed as bundles, both Λ_t^q and Λ_t^p now come with Spin structures $\overset{F_0}{\underset{[0,1]}{\downarrow}}$, $\overset{F_1}{\underset{[0,1]}{\downarrow}}$.

Now, there is a unique homotopy of paths rel endpoints, which, when thought of as a bundle, carries a Spin structure restricting to given ones at various ends.

$\begin{array}{c} F \\ \downarrow \\ [0,1]\times[0,1] \end{array}$



$\rightsquigarrow$ Resolves ambiguity.

Product structures:

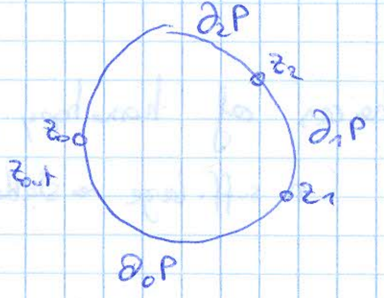
Say we have defined
$$\begin{cases} CF^*(L_0,L_1;H_{L_0,L_1},J_{L_0,L_1}) =: CF^*(L_0,L_1) \\ CF^*(L_1,L_2;H_{L_1,L_2},J_{L_1,L_2}) =: CF^*(L_1,L_2) \\ CF^*(L_0,L_2;H_{L_0,L_2},J_{L_0,L_2}) =: CF^*(L_0,L_2) \end{cases}$$

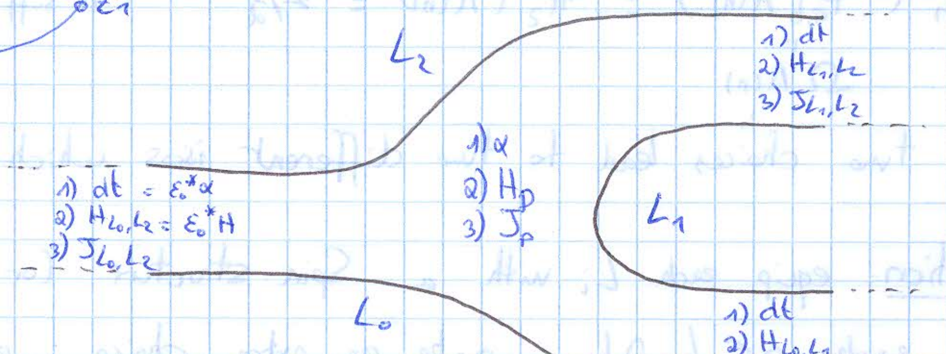
Review the construction of a product map
$$\mu^2: CF^*(L_1,L_2)\otimes CF^*(L_0,L_1)\longrightarrow CF^*(L_0,L_2).$$

Consider $P=\mathbb{D}^2\setminus(3 \text{ boundary points})$, equipped with these "strip-like ends":

$\varepsilon_i^+ : [0,\infty)_s\times[0,1]_t\to P$ around z_i, $i=1$ or 2

$\varepsilon_0^- : (-\infty,0]\times[0,1]\to P$ around z_0.





Equip P with

1) A 1-form α
2) Hamiltonian perturbation term $H_P: P\to C^\infty(X;\mathbb{R})$
3) P-dependent almost complex structure
4) "Lagrangian labels" for ∂^iP,

such that a consistency condition is satisfied near each end, as in the picture.

Rem: in some nice cases, we may be able to eg choose $H_P = 0$ or J_P P-independant (& integrable ??) in order to make computations. But it is not generally possible, eg with $L_0 = L_1 = L_2 = L$.

Now, given $z_2 \in \mathcal{X}^{H_{L_1L_2}}_{L_1L_2}$ (time-1 chords $L_1 \to L_2$, "intersection points"), $z_1 \in \mathcal{X}_{L_0,L_1}$, $z_0 \in \mathcal{X}_{L_0,L_2}$, consider

$$\mathcal{M}(z_{out}; z_1, z_2) = \begin{cases} u : P \to X \\ (*)\ (du - X_{H_P} \otimes \alpha)^{0,1} = 0 \quad \text{wrt } J_P \\ u|_{\partial_i P} \subseteq L_i \\ \lim_{s \to +\infty} ((\varepsilon_i^+)^* u)(s,t) = z_i \\ \lim_{s \to -\infty} ((\varepsilon_0^-)^* u)(s,t) = z_0 \end{cases}$$

Rems: strip-like ends help

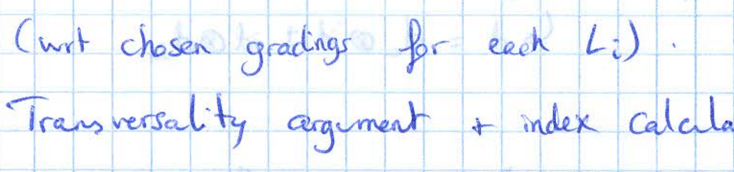
(a) Analytically in defining $W^{k,p}(P, X, L_i$, asymptotics with exponential decay)

(b) in establishing in compactness analysis that when a strip breaks off, the equations solved is <u>exactly</u> Floer's equations for $H_{L_iL_j}$, $J_{L_iL_j}$.

In nice cases (eg, no disc or sphere bubbling), for generic H, J:

(1) $\mathcal{M}(z_0, z_1, z_2)$ is a manifold of dimension $\deg(z_0) - \deg(z_1) - \deg(z_2)$ (wrt chosen gradings for each L_i).

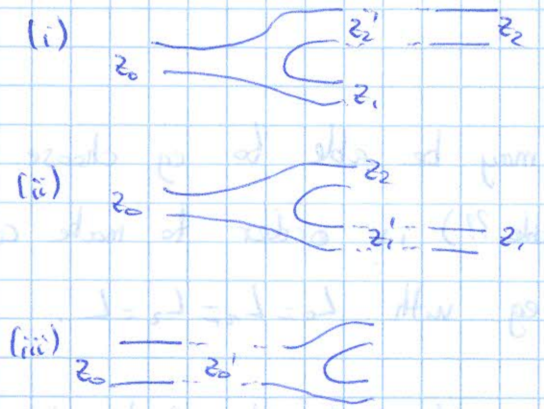
Transversality argument + index calculation + gluing argument of local CR operators



$$\sim \quad \mathrm{ind}(D_u \# z_2) = \underbrace{\mathrm{ind}(D_u)}_{\deg(z_0) - \deg(z_1)} + \underbrace{\mathrm{ind}(D_{z_2})}_{\deg(z_2)}$$

(2) Gromov's compactness + gluing: in the absence of bad bubbles, $\mathcal{M}$ is compactifiable with exactly 3 types of codim-1 boundary

("the energy accumulates near punctures):



The equations we solve in these broken off parts are just ∂_{L_1,L_2}, ∂_{L_0,L_1}, ∂_{L_0,L_2}, thanks to the consistency conditions.

(3) $\mathcal{M}$ is orientable "rel. orientations at its ends", i.e. there exists a canonical isomorphism (if we have fixed Spin structures, etc)

$$\lambda(T\mathcal{M}(z_{out}; z_1, z_2)) \cong \mathcal{O}_{z_0} \otimes \mathcal{O}_{z_1}^\vee \otimes \mathcal{O}_{z_2}^\vee$$

With all of this, we can define

$$\mu^2(z_2, z_1) = \sum_{\substack{z_{out},\, \beta \\ \mathrm{ind}(\beta)=0}} \# \mathcal{M}_2(z_{out}; z_1, z_2, \beta) \cdot T^{\overset{\text{depends on } H_p \text{ but not } u}{E(\beta)}} \cdot z_{out} \cdot (-1)^{\deg z_1}$$

↳ compact 0-dim manifold

↳ see last lecture for how to turn this into a signed count.

Because the signed count of ∂(1D component of $\mathcal{M}(z_{out}; z_1, z_2)$) is 0, we get

Proposition: $\mu^2_{\{H_p,\alpha_p,J_p\}}(z_2, \underbrace{\mu^1(z_1)}_{\partial_{L_0,L_1}}) + (-1)^{\deg z_1 - 1} \mu^2(\underbrace{\mu^1(z_2)}_{\partial_{L_1,L_2}}, z_1) + \underbrace{\mu^1}_{\partial_{L_0,L_2}}(\mu^2(z_2, z_1)) = 0$,

so μ^2 is a chain map $\mu^2 : C_\bullet \otimes D_{\bullet\,?} \to E_\bullet$,

↳ $d = d_C \otimes \mathrm{id} \pm \mathrm{id} \otimes d_D$

and so it descends to

$$[\mu^2_{\{H_p,\alpha_p,J_p\}}] : HF^*(L_1, L_2) \otimes HF^*(L_0, L_1) \to HF^*(L_0, L_2)$$

Lemma ("homotopy of choices") $[\mu^2_{\{H_p,\alpha_p,J_p\}}]$ is independent of the choices of $\{H_p, \alpha_p, J_p\}$ (in absence of bad bubbling).

Theorem [Donaldson] $[\mu^2]$ gives the composition in a category, the Donaldson–Fukaya category $H^0\mathcal{F}uk(X)$ ($= D\mathcal{F}uk(X)$, where the D stands for "Donaldson", not "derived").

Objects: Say $2c_1(X)=0$, and fix a choice of fiberwise universal cover

$$\begin{tikzcd} & \tilde{\mathcal{G}} \arrow[d, "\alpha"] \\ \Lambda(n) \arrow[r] & \mathcal{G} \arrow[d, "\alpha"] \\ & X \end{tikzcd}$$

, or a $\mathbb{Z}/N\mathbb{Z}$ cover.

Definition: a Lagrangian brane is a triple $L^\# = (L, \tilde{\alpha}_L, P)$ where P is a Spin (or rel. Pin) structure, and $\tilde{\alpha}_L$ is a grading

$$\begin{tikzcd} & \tilde{\mathcal{G}}|_L \arrow[d] \\ L \arrow[ur, dashed, "\tilde{\alpha}_L"] \arrow[r, "\alpha_L"'] & \mathcal{G}|_L \end{tikzcd}$$

, or $\mathbb{Z}/N\mathbb{Z}$ grading.

Objects $Ob(H^0\mathcal{F}uk(X)) = \{L^\#\}$

choose some H_{L_0,L_1}, J_{L_0,L_1}.

$$Hom(L_0^\#, L_1^\#) = HF^0(L_0^\#, L_1^\#) = H^0(CF^*(L_0^\#, L_1^\#), \mu^1_{L_0,L_1} = \partial_{L_0,L_1})$$

The composition $Hom(L_1^\#, L_2^\#) \otimes Hom(L_0^\#, L_1^\#) \to Hom(L_0^\#, L_2^\#)$ is given by $[\mu^2](-,-)$.

Need: (a) identity morphisms $[e_L] \in Hom(L,L)$ $\forall L$, identity for $[\mu^2]$.

(b) Composition is associative?

(a) Define $e_L = \sum_{z_0,\beta} \#\mathcal{M}(z_0,\beta)\cdot T^{E(\beta)}\cdot z_0$, where

where $\mathcal{M}(z_{out}) = \begin{cases} u: \mathbb{H} = \mathbb{D}^2\setminus\{-1\} \to X \\ (du - X\otimes\alpha)^{0,1} = 0 \\ u|_{\partial S} \subseteq L \\ \lim_{s\to-\infty} ((\varepsilon_{out}^-)^* u)(s,t) = z_{out}. \end{cases}$ (pick some H, J restricting to $H_{L,L}$, $J_{L,L}$ near neg. end, and α restricts to dt)

We have $ind(u) = deg(z_0)$. In particular, $\mathcal{M}(z_0)$ is only rigid when $deg(z_0) = 0$.

• Compactness + gluing ⇒ codim-1 boundary

$\Rightarrow \partial_{L,L}(e_L) = 0$.

• Why a unit? E.g., $[\mu^2](\overset{CF^*(K,L)\,\ni}{[\alpha]}, [e_L]) = \pm\alpha$.

$\mu^2(\alpha, e_L)$ counts

$\Big\downarrow$ glue + homotopy

~~an operator $I: CF^*(K,L) \circlearrowleft$ counting~~ ~~index 0 solutions~~

Get an operator $I: CF^*(K,L) \circlearrowleft$ counting the index 0 solutions to . The only index 0 solutions are constant

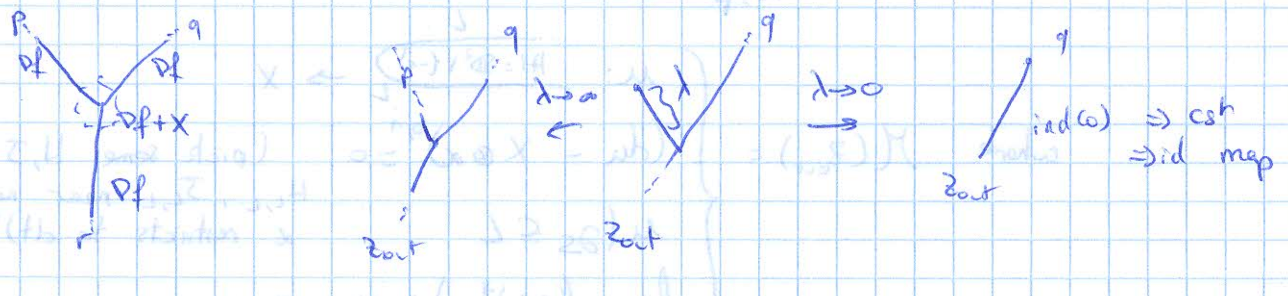
$\Rightarrow I = \pm\mathrm{Id}(\alpha) = \alpha$.

Counting solutions to the parametrized moduli space (by homotopy param.) gives a chain homotopy $\mu^2(\alpha, e_L) - \alpha = \partial K + K\partial$.

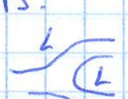
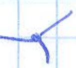
Rem. if $L \subseteq T^*L$ 0-section, then $H = \pi^* f$, $f: L \to \mathbb{R}$. If H is C^2-small, then the associated e_L is $\sum_{\text{local minimum}} p$

$M(p) := \{\gamma: \mathbb{R}_- \to L \mid \dot\gamma = -Df|_\gamma\} = W^u(p)$

In Morse theory, the product is



Proposition [Fukaya-Oh] for $L \subseteq T^*L$ 0-section, $\exists$ canonical choices of grading & rel. Pin structures so that $(HF^*(L,L), [\mu^2]) \cong H^*(L)$, as algebras with units. $\qquad HF^*(L,L) \cong HM^*(L,f) \cong H^*(L)$

The proof compares  to flowlines  for "nice" H, J...