25/04/16 <u>So far</u>: (X^{an}, ω) satisfying $(*)$

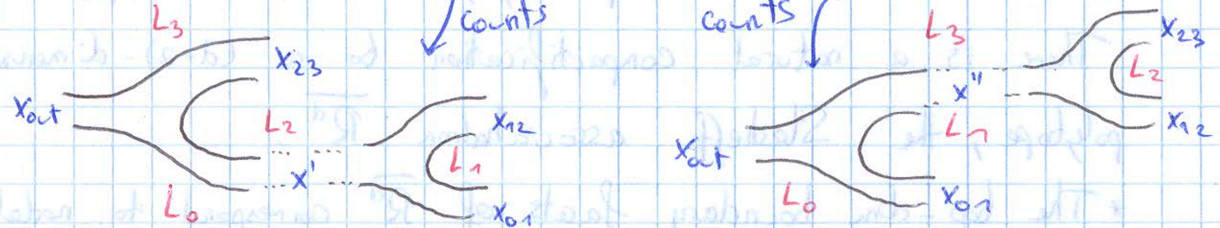
$K, L \subseteq X$ Lagrangians (branes) satisfying $(**)$ $\rightsquigarrow CF^*(K,L) \circlearrowleft \mu^1$ depending on $H_{K,L}$, $J_{K,L}$, but cohomologically independent of choices.

Given (L_0, L_1, L_2) we defined (depending on more choices, compatible with previous choices) a chain map

$$\mu^2 : CF^*(L_1, L_2) \otimes CF^*(L_0, L_1) \to CF^*(L_0, L_2)$$

st $[\mu^2]$ is independent of choices. This nearly gives a "categorical composition law" in $H^0 Fuk(X)$, <u>but</u> is it associative? On cohomology level yes, but not on the chain level.

For $L_0 \xrightarrow{x_{01}} L_1 \xrightarrow{x_{12}} L_2 \xrightarrow{x_{23}} L_3$, on the chain level, there is no reason for $\mu^2(x_{23}, \mu^2(x_{12}, x_{01}))$ to equal $\mu^2(\mu^2(x_{23}, x_{12}), x_{01})$:



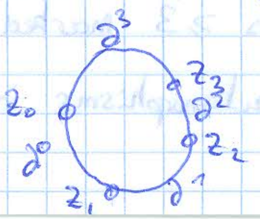
But we will show that the difference

$$\mu^2(x_{23}, \mu^2(x_{12}, x_{01})) - \mu^2(\mu^2(x_{23}, x_{12}), x_{01})$$

is null homotopic*, so $[\mu^2]$ is associative. * via a "geometric null homotopy"

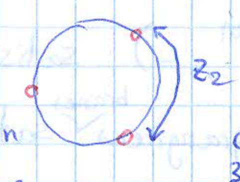
We will deduce this as a special case of constructions of "higher A_∞ homotopies".

Let R^3 denote the space of discs with $3+1=4$ (inputs + output) marked points removed from the boundary, modulo automorphisms. A representative:

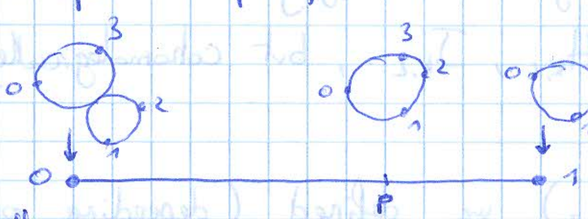


Up to biholomorphism, the position of 3 of the points (eg z_0, z_1, z_3) can be fixed at $-e^{2i\pi\frac{k}{3}}$.

So, $\mathcal{R}^3 \underset{\text{diff}}{\cong} (0,1)$:



There is a natural "Deligne Mumford" compactification of $\mathcal{R}^3$ to $\overline{\mathcal{R}^3} = [0,1]$:



(only 1 such picture, because ≥ 3 pts on bubble.

Notice that over 0 and 1, it looks like the pictures we drew for $\mu^2(-,\mu^2(-,-))$ and $\mu^2(\mu^2(-,-),-)$.

More generally, let $\mathcal{R}^d$ $(d \geq 2)$ be the space of discs with $d+1$ boundary marked points removed, labeled $\underbrace{z_0}_{\text{output}}, \underbrace{z_1, \dots, z_d}_{\text{inputs}}$ cyclically ordered counterclockwise. On each $S \in \mathcal{R}^d$, there is an induced decomposition $\partial S = \coprod_i \partial^i S$, where $\partial^i S$ is between z_i and z_{i+1} mod $d+1$.

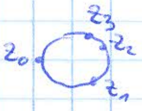
Up to biholomorphism, we can fix the positions of z_0, z_1, z_d.
$\Rightarrow \dim \mathcal{R}^d = d-2$. Explicitly, $\mathcal{R}^d = \mathrm{Conf}_{d+1}(\partial D^2) / \mathrm{Aut}(D^2)$.
There is a natural compactification to a $(d-2)$-dimensional polytope, the Stasheff associahedron $\overline{\mathcal{R}^d}$.

* The top-dim boundary facets of $\overline{\mathcal{R}^d}$ correspond to nodal degenerations

$D \rightsquigarrow D_1 \cup D_2$
(D: disks; $D_1 \cup D_2$: pair of disks, each carries ≥ 3 marked points, including nodes)

There is a corresponding universal family of domains $\overline{\mathcal{S}^d} \to \overline{\mathcal{R}^d}$ (repr. of Riemann surface); call S_r the fiber over $r \in \overline{\mathcal{R}^d}$. (⚠ the fiber is the point r, as top. space)

ex: $\overline{\mathcal{R}^3} \cong [0,1]$, $S_p =$

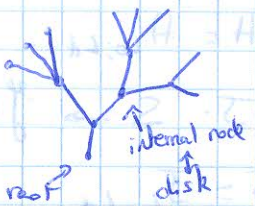


, $S_0 =$

S_r is represented by a potentially nodal Riemann surface of the form

each component has ≥ 3 marked points (stability), mod automorphisms in each component.

There is an underlying combinatorial type of a tree:



$\text{codim}\,(\overline{R}^d_T) = \#$ interior nodes of the tree.
(T = tree)

Now, we are going to put strip-like ends. Recall

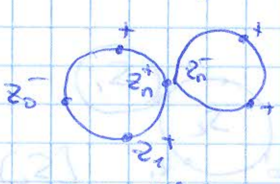
$$Z_+ := [0,1] \times [0,\infty)$$
$$Z_- := [0,1] \times (-\infty, 0]$$

Definition: a <u>choice of strip-like ends</u> is, for each $i \in \{1,\dots,d\}$ and for $r \in R^d$: $\varepsilon_i^+ : Z_+ \to S_r$, around z_i,

$\varepsilon_0^- : Z_- \to S_r$, around z_0

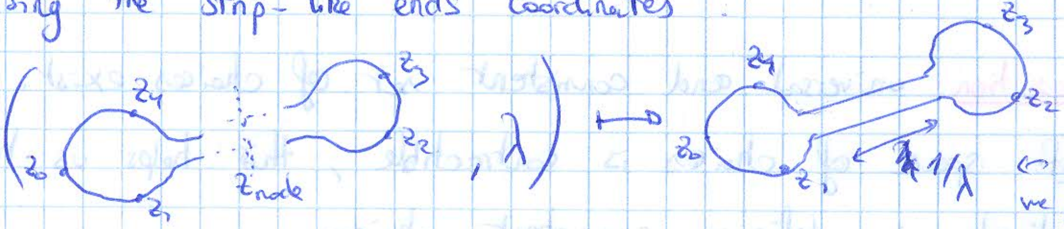
varying smoothly in r.

If $r \in \overline{R^d}$: $\varepsilon^\pm : Z_\pm \to S_r$ around each interior node, in a manner dictated by the picture:



Limits of choices should agree with choices made.

Given such a choice, note that I obtain a gluing map $(\partial \overline{R}^d)_T \times [0,\infty)^{\#\text{inter}} \longrightarrow \overline{R}^d$, given by connect summing using the strip-like ends coordinates



$\leftarrow$ so that when $\lambda \to 0$, we get the degenerate thing.

$\Rightarrow \overline{R}^d$ is a manifold with corners.

Definition: fix a consistent family of strip-like ends on $\overline{R}^d$, for all d (inductively). A <u>universal and consistent choice of Floer data</u> is a smoothly varying choice, inductively for each d and each d-tuple $(L_0,\dots,L_d)$ of objects ("Lagrangian branes"), for each $s \in \overline{R}^d$, of:

(a) Hamiltonian term $H: S \to C^\infty(X;\mathbb{R})$ with

$$(\varepsilon_i^+)^* H = H_{L_i, L_{i+1}} \quad ; \quad (\varepsilon_0^-)^* H = H_{L_0, L_d}$$

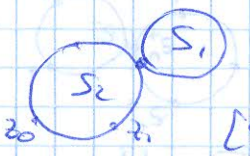
(b) Tame / compatible almost complex structure $J : S \to \mathcal{J}(X)$ with

$$(\varepsilon_i^+)^* J = J_{L_i, L_{i+1}} \quad ; \quad (\varepsilon_0^-)^* J = J_{L_0, L_d}$$

(c) 1-form α on S with $(\varepsilon_i^\pm)^* \alpha = dt$

This choice should be

* smoothly varying in S
* consistent, meaning that the restriction of this data to a component S of a corner stratum with induced suitable labelings $(L_{i_1}, \dots, L_{i_k})$ agrees with choices already made for this tuple $(L_{i_1}, \dots, L_{i_k})$ and $S \in \mathcal{R}^{d_i}$, $d_i < d$, and that these choices vary smoothly across the corner charts.



$[S_1] \in S^2$

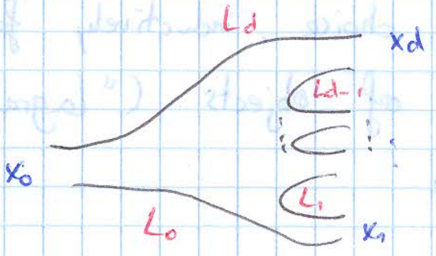
$[S_2] \in S^3$ $\rightsquigarrow [S_1 \#_{z_0, z_2} S_1] \in S^4$

So, the choices of (a), (b), (c) for internal vertices of the tree are determined by this consistency condition.

Proposition: universal and consistent set of choices exist.

Proof: space of choices is contractible ; this helps us to proceed inductively in defining consistent choices. $\square$

Make such a choice. Given a tuple $(L_0, \dots, L_d)$ and chords / intersection points $x_i \in \mathcal{X}_{L_i, L_{i+1}}^{H_{L_i, L_{i+1}}}$, $x_0 \in \mathcal{X}_{L_0, L_d}^{H_{L_0, L_d}}$, get $\mathcal{R}^d(x_0; x_d, \dots, x_1)$:



$$\mathcal{R}^d(x_0; x_d, \dots, x_1) = \begin{cases} r \in \mathcal{R}^d, \quad u : S_r \to X \\ u(\partial^i S) \subseteq L_i \\ \lim\limits_{s \to +\infty} ((\varepsilon_i^+)^* u)(s,t) = x_i \\ \lim\limits_{s \to -\infty} ((\varepsilon_0^-)^* u)(s,t) = x_0 \\ (du - X_{S_r} \otimes \alpha_{S_r})^{0,1} = 0 \end{cases}$$

(X_{S_r}: Ham. v.f. for H_{S_r}; α_{S_r}: 1-form; $(\,)^{0,1}$: wrt J_{S_r})

Remarks.

(i) Transversality and dimension: for generic choices, in the absence of "bad" bubbling, $\mathcal{R}^d(x_0; x_d, \dots, x_1)$ is a manifold of dimension

$$\underbrace{d-2} + \deg(x_{out}) - \sum_{i=1}^{d} \deg(x_i)$$

($d-2$ comes from $\dim \mathcal{R}^d$; assuming L_i are graded)

(in degenerate case $d=1$, i.e. strip, this agrees with $\dim \left(\mathcal{M}(x_{out}, x_1)/\mathbb{R}\right)$, where $\mathcal{R}^1(x_{out}; x_1) = \mathcal{M}(x_{out}, x_1)/\mathbb{R}$).

(ii) $\mathcal{R}^d(x_0; x_d, \dots, x_1)$ is canonically oriented rel. ends and a choice of orientation of $\mathcal{R}^{d-2}$:

$$\lambda(T\mathcal{R}^d(x_0; x_1, \dots, x_d)) \cong \lambda(T\mathcal{R}^d) \otimes o_{x_0} \otimes o_{x_1}^\vee \otimes \dots \otimes o_{x_d}^\vee .$$

(iii) Compactness + gluing: the limit configurations allowed by Gromov compactness are:

- bubbling of spheres / discs, excluded by (*), (**) (assumptions in the beginning).
- bubbling of strips at marked points:

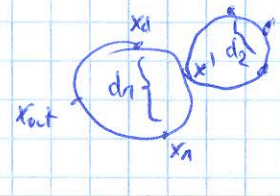
- degeneration of domain to $\partial \overline{\mathcal{R}^d}$:

$\Rightarrow$ Gromov-Floer compactification $\overline{\mathcal{R}^d(x_{out}; x_1, \dots, x_d)}$, which in codim 1 is covered by the images of the natural inclusions of

$\overline{\mathcal{R}^{d_1}(x_0; \dots)} \times \overline{\mathcal{R}^{d_2}(x_1', \dots)}$

$d_1, d^2 \geq 1$

$d_1 + d_2 - 1 = d.$



Define $\mu^d : CF^*(L_{d-1}, L_d) \otimes \dots \otimes CF^*(L_0, L_1) \longrightarrow CF^*(L_0, L_d)$:

$$\mu^d(x_d, \dots, x_1) = \sum_{\substack{x_{out} \in \mathcal{X}_{L_0, L_d} \\ \deg(x_{out}) = (\sum \deg x_i) + 2 - d \\ \beta \in \pi_2(X; \{L_i\})}} (-1)^{*_d} \# \mathcal{R}^d(x_0; x_1, \dots, x_d, \beta) \cdot T^{E(\beta)} \cdot x_{out}$$

where $\mathcal{R}^d(x_0; x_1, \dots, x_d, \beta)$ is signed as before, and $*_d = \sum_{i=1}^d i \deg(x_i)$,

and $\mathcal{R}^d(x_0; x_1, \dots, x_d) = \coprod_{\beta \in \pi_2(X, \dots)} \mathcal{R}^d(x_0; x_1, \dots, x_d, \beta)$.

Proposition: $\sum (-1)^{*_i} \mu^{d-k+1}(x_d, \dots, x_{i+k+1}, \mu^k(x_{i+k}, \dots, x_{i+1}), x_i, \dots, x_1) = 0$.

$*_i = \sum_{j=1}^{i} \deg x_j - 1$ $\qquad A_\infty$-equations.

Proof: by picture and compactness analysis.

First few equations:

- $d=1$: $\mu^1 \circ \mu^1 = 0$, ie μ^1 is a differential.
- $d=2$: $\mu^2(\mu^1 \otimes id) \pm \mu^2(id \otimes \mu^1) + \mu^1 \circ \mu^2(-,-) = 0$, ie μ^2 is a chain map, so descends to cohomology wrt μ^1.
- $d=3$: $\mu^1 \mu^3(x_3, x_2, x_1) \pm \mu^3(\mu^1(x_3), x_2, x_1) \pm \mu^3(x_3, \mu^1(x_2), x_1) \pm \mu^3(x_3, x_2, \mu^1(x_1))$

$$= \underbrace{\pm \mu^2(x_3, \mu^2(x_2, x_1)) \mp \mu^2(\mu^2(x_3, x_2), x_1)}_{\text{associator}}.$$

So μ^3 is a chain homotopy between this associator and 0, as desired.

- higher homotopies...

Note: μ^k $(k>2)$ don't descend to H^* !

Theorem: $(\mathcal{F}uk(X), \mu^\bullet)$ is an invariant of X up to quasi-isomorphism (next time).