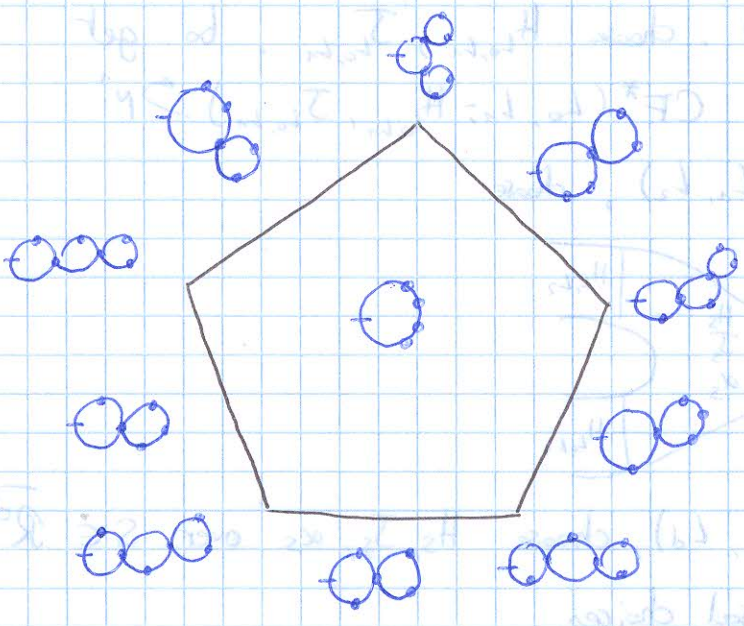
27/04/16: Here is a picture of the Stasheff polytope $\overline{\mathcal{R}}^4$:



Let Λ be a field.

Definition: a ($\mathbb{Z}$-graded) A_∞-category (over Λ) $\mathcal{C}$ consists of the following data:

* a set of objects $\mathrm{ob}\,\mathcal{C}$
* for any $K, L \in \mathrm{ob}\,\mathcal{C}$, a $\mathbb{Z}$-graded vector space $\hom^\bullet_{\mathcal{C}}(K,L)$
* for $k \geq 1$ and any $(k+1)$-tuple $(L_0, \dots, L_k)$ in $\mathrm{ob}\,\mathcal{C}$, "higher composition map":

$$\mu^k : \hom_{\mathcal{C}}(L_{k-1}, L_k) \otimes \dots \otimes \hom_{\mathcal{C}}(L_0, L_1) \to \hom_{\mathcal{C}}(L_0, L_k)$$

of degree $2-k$, satisfying A_∞-relations: for each $d \geq 1$,

$$\sum_{i,s} (-1)^{*_i} \mu^{d-s+1}(x_d, \dots, x_{s+i+1}, \mu^s(x_{s+i}, \dots, x_{i+1}), x_i, \dots, x_1) = 0.$$

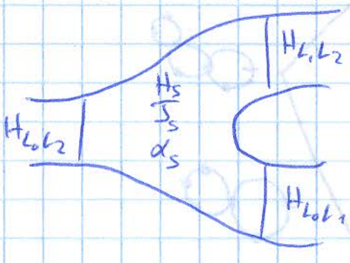
We showed:

Theorem: for (X^{2n}, ω) symplectic satisfying $(*)$ (and maybe $2c_1(X) = 0$ and fixed $\tilde{\mathcal{L}} \to \mathcal{L} \to X$), we can make choices to define an A_∞ cat. over Λ whose objects are Lagrangian branes $L = (L, \tilde{\alpha}_L, P)$ satisfying $(**)$. ~~be~~ ~~get~~ called $\mathcal{F}uk_S(X)$

$\uparrow$ set of choices

What choices?

* for each (L_0, L_1), choose H_{L_0,L_1}, J_{L_0,L_1}, to get $\hom_{\mathcal{F}_S}(L_0, L_1) := CF^*(L_0, L_1; H_{L_0,L_1}, J_{L_0,L_1}) \circlearrowleft \mu^1$
* for each (L_0, L_1, L_2), chose



* for each $(L_0, \dots, L_d)$, choose H_S, J_S, α_S over $S \in \overline{\mathcal{R}}^d$, consistent universal choices.

Rem. A_∞ categories do not induce ordinary categories (ie ~~...~~ no forgetful functor), before taking H^0.

Definition: an A_∞-functor $\mathcal{F} : \mathcal{E} \to \mathcal{D}$ between A_∞-categories is the following data:

* a map $\mathcal{F} : \mathrm{ob}\, \mathcal{E} \to \mathrm{ob}\, \mathcal{D}$
* for all $d \geq 1$ and $(L_0, \dots, L_d)$ in $\mathrm{ob}\, \mathcal{E}$, a linear map
$\mathcal{F}^d : \hom_{\mathcal{E}}(L_{d-1}, L_d) \otimes \dots \otimes \hom_{\mathcal{E}}(L_0, L_1) \to \hom_{\mathcal{D}}(\mathcal{F}L_0, \mathcal{F}L_d)$
of degree $1-d$ satisfying the A_∞-functor relations for each d and $x_d, \dots, x_1$ composable morphisms in $\mathcal{E}$,

$$\sum (-1)^{*_i} \mathcal{F}^{d-s+1}(x_d, \dots, x_{i+s+1}, \mu^s_{\mathcal{E}}(x_{i+s}, \dots, x_{i+1}), x_i, \dots, x_1)$$
$$= \sum_{\ell, i_1, \dots, i_\ell} \mu^\ell_{\mathcal{D}}(\mathcal{F}^{i_1}(x_d, \dots), \mathcal{F}^{i_2}(\dots), \dots, \mathcal{F}^{i_\ell}(\dots, x_1))$$

Let us examine that

* $d=1$: $\mathcal{F}^1 \circ \mu^1_{\mathcal{E}} = \mu^1_{\mathcal{D}} \circ \mathcal{F}^1$, i.e. $\mathcal{F}^1$ is a chain map.
* $d=2$: $\mathcal{F}^1 \mu^2_{\mathcal{E}}(x_2, x_1) - \mu^2_{\mathcal{D}}(\mathcal{F}^1(x_2), \mathcal{F}^1(x_1)) = \pm \mathcal{F}^2(\mu^1_{\mathcal{E}}(x_2), x_1) + \mathcal{F}^2(x_2, \mu^1_{\mathcal{E}}(x_1))$
$+ \mu^1_{\mathcal{D}} \mathcal{F}^2(x_2, x_1)$,

i.e. $[\mathcal{F}^1]$ intertwines compositions $[\mathcal{F}^1][\mu^2_{\mathcal{E}}](\alpha_2, \alpha_1) = [\mu^2_{\mathcal{D}}]([\mathcal{F}^1]\alpha_2, [\mathcal{F}^1]\alpha_1)$.

Rem: F A_∞ functor induces $[F^1] : H^0\mathcal{E} \to H^0\mathcal{D}$ a honest functor between honest categories (except maybe they don't have a unit).

A functor $F : \mathcal{E} \to \mathcal{D}$ is cohomologically full and faithful if

$$[F^1] : H^*(\hom_{\mathcal{E}}(K,L)) \xrightarrow{\sim} H^*(\hom_{\mathcal{D}}(K,L)) \quad \forall K, L \in \mathrm{ob}\,\mathcal{E}$$

F is a quasi-isomorphism if F is cohomologically full and faithful and $F : \mathrm{ob}\,\mathcal{E} \to \mathrm{ob}\,\mathcal{D}$ is a bijection.

Rem: a better notion is a quasi-equivalence: $F : \mathcal{E} \to \mathcal{D}$ cohomologically full and faithful, st $[F^1] : H^0\mathcal{E} \to H^0\mathcal{D}$ is essentially surjective, ie every object is isomorphic to an object in the image.

Two objects X, Y in a graded category $\mathcal{C}$ are isomorphic if $\exists$ $f \in \mathrm{Hom}^0(X,Y)$, $g \in \mathrm{Hom}^0(Y,X)$ with $f \circ g = id_X$ and $g \circ f = id_Y$.

Theorem: $X^{an} \rightsquigarrow Fuk_S(X)$. Different choices lead to quasi-isomorphic A_∞-categories

$\hookrightarrow$ get $Fuk(X)$, well-defined up to quasi-isomorphic.

Rem: if $\mathcal{E}$ has just one object L, the data of an A_∞-category reduces to a graded vector space $A := \hom^*_{\mathcal{E}}(L,L)$ and for any $d \geq 1$, $\mu^d : A^{\otimes d} \to A$ of degree $2-d$ satisfying the A_∞-relations.

$\longrightarrow$ A_∞-algebra A.

Why A_∞-algebras/categories?

examples: if A is an A_∞-algebra with $\mu^d = 0$ $\forall d > 2$, we get a differential graded algebra $(A, d = \mu^1, \cdot = \mu^2)$.

Many examples: Y top. space $\rightsquigarrow C^*(Y)$ singular cochains is a DGA.

It is well understood in topology that the passage $C^*(Y)$ (DGA) to $H^*(Y)$ (assoc algebra) looses lots of information.

(54)

ex: (Massey products) let A be a DGA.

If a, b, c are cocycles in A with $[a \cdot b] = [b \cdot c] = 0$, then we can define a homology class in $H^*(A)$, the Massey product of a, b, c, as follows:

- choose τ with $d\tau = a \cdot b$
- choose κ with $d\kappa = b \cdot c$
- set $\langle a, b, c \rangle = \tau \cdot c \mp a \cdot \kappa$

Check: d (this class) $= 0$ and cohomologically, the result is independant of choices (up to adding $[a]$, $[c]$).

Point: "the triple product $[a \cdot b \cdot c]$ is zero for 2 different reasons", and the "sum of these reasons" is a secondary class.

There exists also higher Massey products for n-uples $(a_1, \dots, a_n)$, with all $n-k$ Massey products of subsequences vanish for all $k \geq 1$.

ex: B = Borromean rings in S^3. We can check that as algebras, $H^*(S^3 \setminus B) \cong H^*(S^3 \setminus$ 3 copies of unknot, unlinked$)$, BUT there are non trivial Massey products on $H^*(S^3 \setminus B) \cong H_{3-*}(S^3, B)$ on x_1, x_2, x_3 in degree 1 Poincaré dual to the standard bounding disks.
$\longrightarrow$ "higher linking".

We would like to retain that information. One use of A_∞ structures:

Theorem: ["Homological Perturbation Lemma", "Transfer Theorem"] Given a DGA A (or A_∞-algebra A), there is an A_∞-structure on $H^*(A, d)$ (with $\mu^1 = 0$), which is quasi-isomorphic to the original A. "B"

Unfortunately, this A_∞-structure (B, μ_B) on B is not unique, for instance given any D with a sequence $F^d : D^{\otimes d} \longrightarrow B$ with F^1 an isomorphism, we can pull back $F^* \mu_B$ to get an A_∞-structure on D, and

$\{F^d\}$ induces a quasi-isomorphism $(D, F^*\mu_B^\bullet) \xrightarrow{\sim} (B, \mu_B^\bullet)$

$(B, \mu_B^\bullet)$ knows about all Massey products in A!

(Forgot: a morphism $F: C \to D$ is maps $F^d: C^{\otimes d} \to D$, $d \geq 1$, satisfying the functor equations).

Also, $\mathrm{rk}\, B$ is strictly smaller than $\mathrm{rk}\, A$.

Other advantages of A_∞-algebras:

Rational homotopy theory: says that the "rational homotopy type" of a space Y is determined "up to quasi-isomorphism of DGAs" by $(C^\bullet(Y;\mathbb{Q}), d, \smile)$

Problem: ordinarily, define a map of DGAs to be a chain map $F: A \to B$ intertwining $\smile_A, \smile_B$ strictly, and F is a quasi-iso if $[F]: H^\bullet A \xrightarrow{\simeq} H^\bullet B$. But a quasi-isomorphism of DGAs is NOT an equivalence relation, as quasi-isos of DGAs are not invertible.

↳ "Theorem": X, Y have the same rational homotopy type if, roughly, $\exists$ zig-zag of DGA quasi-isos

$$C^\bullet(X;\mathbb{Q}) \xrightarrow{\sim} D_1 \xleftarrow{\sim} D_2 \xrightarrow{\sim} \cdots \xleftarrow{\sim} C^\bullet(Y;\mathbb{Q})$$

This problem disappears in the A_∞ setting:

Theorem: A_∞ quasi-isomorphisms are invertible (up to homotopy): given $F: A \xrightarrow{\simeq} B$ A_∞-quasi-iso, $\exists$ $G: B \xrightarrow{\simeq} A$ with $F \circ G \sim id_A$ and $G \circ F \sim id_B$.

(may require working over a field, or being very careful about working over projectives)

Proof:

$$\begin{tikzcd} A \arrow[d, "\wr"] & & B \arrow[d, "\wr"] \\ H^\bullet A \arrow[rr, "\text{easy to invert}", "\text{by recursion}"'] & & H^\bullet B \arrow[ll, bend left] \end{tikzcd}$$

Hom. pert. lemma.

because no differential. ☐

Rem: a morphism of A_∞-algebras is $\{F^d : A^{\otimes d} \to B\}$. Get $F = \oplus F^d : TA \to B$, where TA is the bar complex of A.

Definition: A (A_∞/DGA) is formal if

$$A \underset{A_\infty \text{ quasi-iso}}{\simeq} (H^\bullet A,\ d = 0,\ \cup = [\mu^2],\ \mu^{\geq 3} = 0)$$

($\Rightarrow$ no Massey products)

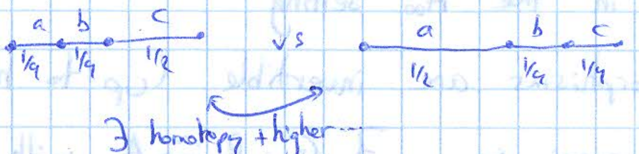
($\Leftarrow$ coincides with the notion of DGA formality: a DGA B is formal if $B \xrightarrow{\sim} \cdot \xleftarrow{\sim} \cdot \xrightarrow{\sim} \cdots \xleftarrow{\sim} (H^\bullet B, \cup)$, with DGA quasi-isos).

ex: Theorem [Deligne - Griffiths - Morgan - Sullivan] if X Kähler and $\pi_1 X = 0$, then $C^\bullet(X; \mathbb{Q})$ is formal

(Hodge theory constructs a 2 steps zig-zag).

Other classical appearance of A_∞-structures:

Stasheff: ΩY, with composition $\Omega Y \times \Omega Y \xrightarrow{\text{concatenation}} \Omega Y$ is an "A_∞-space". So $C_*(\Omega Y)$ is an A_∞-algebra



Back to Floer theory:

$L \hookrightarrow T^*L$ 0-section; equip with canonical brane structure (if L spin)

$\rightsquigarrow$ $(CF^*(L,L) = \hom_{Fuk(T^*L)}(L,L), \mu^\cdot)$ A_∞-algebra, well-defined up to quasi-isomorphism.

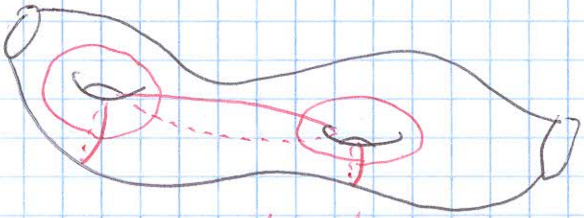
Rem: can define over $\mathbb{C}$ or $\mathbb{Z}$! (Because exact, so $A_H : \mathcal{P}_{L,L} \to \mathbb{R}$, and not a cover), (weigh all discs by 1).

Exactness tells us that the count is still finite, because the action of asymptotic chords gives a priori energy bounds.

Theorem [Fukaya–Oh rationally, Abouzaid over $\mathbb{Z}$] $\exists$ A_∞-equivalence $CF^*(L,L) \simeq C^\cdot(L)$, so $CF^\cdot(L,L)$ is formal iff L is.

Examples in Riemann surfaces:

$X_1 = T^2 \setminus pt$



$X_2 = \Sigma_2 \setminus 2$ pts

Theorem [Lekili–Perutz] the Fukaya category with objects (L_1, L_2) is not formal

Theorem [Seidel] ... objects $(L_1, \dots, L_5)$ is not formal.

In either cases, no visible disks, but "small discs" appear when we perturb.

Punchline: even these Fukaya categories are not combinatorially computable to all orders.