(58)

02/05/16

Today: miscellaneous topics.

1) Units in A_∞-categories / the Fukaya category
2) Local systems on Lagrangians
3) Cases in which L bounds discs, X has spheres, but things still work simply (monotone).

Units.

There are several possible notions of "units" / "identity morphisms" in an A_∞-category $(\mathcal{C}, \mu^\cdot)$.

Terminology: $\mathrm{Hom}^*_{\mathcal{C}}(A,B) := H^* \hom_{\mathcal{C}}(A,B)$.

Definition: an A_∞-category $\mathcal{C}$ is cohomologically unital ("c-unital", "h-unital") if, for every $X \in ob(\mathcal{C})$, $\exists\ [e_X] \in \mathrm{Hom}^0_{\mathcal{C}}(X,X)$ satisfying
$$\left[\mu^2(e_X, \sigma)\right] = (-1)^{\deg(\cdot)}[\mu^2(\sigma, e_Y)] = [\sigma], \quad [\sigma] \in \mathrm{Hom}^*(Y,X).$$

Let (X^{2n}, ω), satisfying $(*)$.

[**Proposition:** $Fuk(X)$ is c-unital.

Proof: already given, when talking about Donaldson Fukaya category. $\square$

There is a stricter notion which is frequently convenient.

Definition: an A_∞-category $\mathcal{C}$ is strictly unital if $\forall X \in ob(\mathcal{C})$, $\exists$ morphisms $e_X^+ \in \hom^0(X,X)$ with

* $\mu^2(e_X^+, \sigma) = (-1)^{\deg(\cdot)} \mu^2(\sigma, e_Y^+) = \sigma$, $\quad \forall \sigma \in \hom^*(Y,X)$.
* $\mu^k(\ldots, e_X^+, \ldots) = 0$ for $k > 2$. $\quad \cdot\ \mu^1(e_X^+)$.

A priori, the geometric elements that we have produced $[e_L] \in HF^0(L,L)$ are not strict units. We need 2 choices for producing strict units:

- Use $[e_X]$ + higher compatibilities/homotopies to produce a homotopy unit ("keeps track of all homotopies" [Fukaya, FOOO]

$\rightsquigarrow$ strictly unital category $\mathcal{F}_h(X) \overset{q.i.}{\simeq} \mathcal{F}(X)$

$\uparrow$ Fukaya cat. with homotopy units.

- **Proposition** [Lefevre, Seidel] any c-unital A_∞-category $\mathcal{E}$ is quasi-iso to $\tilde{\mathcal{E}}$ strictly unital; we can assume $\tilde{\mathcal{E}}$ is minimal (ie $\mu^1 = 0$).

So, up to quasi-iso,

c-unital $\Leftrightarrow$ strictly unital $\Leftrightarrow$ homotopy unital.

Local systems:

There is an enlargement of the Fukaya category whose objects are Lagrangian branes equipped with local systems with flat connections [Kontsevich].

Given $\begin{matrix}\mathcal{E}_K\\ \downarrow \\ K\end{matrix}$, $\begin{matrix}\mathcal{E}_L\\ \downarrow \\ L\end{matrix}$ $\mathbb{R}$-local systems with flat connections, we can define

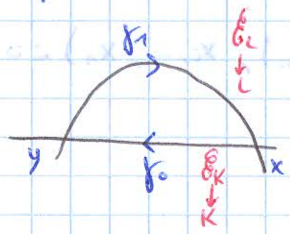
$$\hom_{\mathcal{F}}\left(\begin{matrix}\mathcal{E}_K\\ \downarrow \\ K\end{matrix}, \begin{matrix}\mathcal{E}_L\\ \downarrow \\ L\end{matrix}\right) = CF^\circ\left(\begin{matrix}\mathcal{E}_K\\ \downarrow \\ K\end{matrix}, \begin{matrix}\mathcal{E}_L\\ \downarrow \\ L\end{matrix}\right) = \bigoplus_{\substack{x\in \mathcal{X}_{K,L}\\ \text{time-1 chord}}} \overbrace{\mathrm{Hom}_{\mathbb{R}}\left((\mathcal{E}_K)_{x(1)}, (\mathcal{E}_L)_{x(0)}\right)}^{\text{"x-component"}}$$

If $K \pitchfork L$, we can chose $H_{K,L} = 0$, the x-component is just $\mathrm{Hom}_{\mathbb{R}}\left((\mathcal{E}_K)_p, (\mathcal{E}_L)_p\right)$.

Given $y \in \mathcal{X}_{K,L}$ and $\Phi_x \in \hom_{\mathbb{R}}\left((\mathcal{E}_K)_{x(1)}, (\mathcal{E}_L)_{x(0)}\right)$, $\mu^1(\Phi_x)$ has "y-component"

$$:= \sum_{\substack{\beta,u\in \mathcal{M}(x,y,\beta)/\mathbb{R}\\ \mathrm{ind}(\beta)=1}} T^{E(\beta)}\cdot \mathrm{sgn}(u)\cdot \mathrm{hol}_{[\partial u]}(\Phi_x), \text{ where}$$

$$\mathrm{hol}_{[\partial u]}(\Phi_x) := \mathrm{hol}_{\gamma_0} \circ \Phi_x \circ \mathrm{hol}_{\gamma_1} \in \mathrm{Hom}\left((\mathcal{E}_K)_{y(0)}, (\mathcal{E}_L)_{y(1)}\right).$$



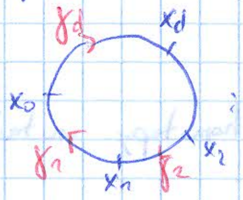
The earlier cases embed into this more general situation by $\begin{matrix}\mathbb{R}\\ \downarrow \\ L\end{matrix}$.

Similarly,

$\mu^d(\Phi_d, \dots, \Phi_1)$ has "x_0 component"

"x^d component" "x^1-component"

$$\sum_{\substack{\beta,\, u \in \bar{\mathcal{R}}^{d}(x_0;\, x_d, \dots, x_1, \beta) \\ \mathrm{ind}(\beta)=0}} T^{E(\beta)} \cdot \mathrm{sgn}(u) \cdot \mathrm{hol}_{[\partial u]}(\Phi_d, \dots, \Phi_1),$$

where: 

; $\mathrm{hol}_{[\partial u]}(\dots) = \mathrm{hol}_{\gamma_0} \circ \Phi_{x_1} \circ \mathrm{hol}_{\gamma_2} \circ \dots \circ \Phi_{x_d} \circ \mathrm{hol}_{\gamma_d}$.

Example: $T^n \subseteq X$ torus supports a n-dim space of rank 1 local systems $\hom(\pi_1(T^n), \mathcal{U}(1))$. They potentially have different $HF^*(\dots)$.

When things still work:

In some cases when X has spheres and/or L has disks, things still work using classical methods.

Meaning: can count strips ($\sim \mu^1$), whose only failure to square to 0 is something we can count.

General structure: a curved/obstructed A_∞-category $\mathcal{C}$ is the data:

* $ob(\mathcal{C})$ as before, $\hom^*(X,Y)$ graded vector spaces
* $\mu^1, \dots, \mu^k, \dots$ as before, but also
* $\mu^0_X \in \hom^2(X,X)$ satisfying curved A_∞-equations

$$\sum_{\substack{i,s \\ s\geq 0}} (-1)^{\maltese_i} \mu^{d-s+1}(x_d, \dots, x_{i+s+1}, \mu^s(x_{i+s}, \dots, x_{i+1}), x_i, \dots, x_1) = 0,$$

where we can have μ^0 insertions.

(i) $\mu^1 \mu^0 = 0$

(ii) $\mu^1(\mu^1(-)) = \mu^2(\mu^0, -) \mp \mu^2(-, \mu^0)$

in some cases, these 2 contributions cancel!

(iii) higher ...

These μ^0 are supposed to "count disk bubbles".

Definition: (X^{an}, ω) is (positively) monotone if $[\omega] = \tau\, c_1(X)$, $\tau > 0$.
$L \subseteq (X^{an}, \omega)$ is monotone if $[\omega] = \lambda\, [\mu]$ $\lambda > 0$, where we think of $[\omega], [\mu] : \pi_2(X, L) \longrightarrow \mathbb{R}$.

ex: • $(\mathbb{CP}^n, \omega_{FS})$

• Fano variety: X complex projective variety with $-K_X$ ample, with $i^*\omega_{FS}$ coming from $X \hookrightarrow \mathbb{P}^N$ with $i^*\mathcal{O}(1) = (K_X)^{-\otimes s}$.

ex:

monotone

not monotone: 2 disks of $\neq$ areas, although same Maslov index.