These μ^0 are supposed to "count disk bubbles".

Definition: (X^{an}, ω) is (positively) monotone if $[\omega] = \tau c_1(X)$, $\tau > 0$.
$L \subseteq (X^{an}, \omega)$ is monotone if $[\omega] = \lambda [\mu]$ $\lambda > 0$, where we think of $[\omega], [\mu] : \pi_2(X, L) \longrightarrow \mathbb{R}$.

ex: • $(\mathbb{CP}^n, \omega_{FS})$

• Fano variety: X complex projective variety with $-K_X$ ample, with $i^*\omega_{FS}$ coming from $X \overset{i}{\hookrightarrow} \mathbb{P}^N$ with $i^*\mathcal{O}(1) = (K_X)^{-\otimes 5}$.

ex:

monotone

not monotone: 2 disks of $\neq$ areas, although same Maslov index.

04/05/16

Monotonicity.

(X^{2n}, ω) is positively monotone if $[\omega] = \tau . c_1(X)$ for $\tau > 0$. (on S^2 classes)

A Lagrangian is monotone if $[\omega] = \lambda [\mu]$ as homomorphisms $\pi_2(X, L) \to \mathbb{R}$. $(\lambda > 0)$

symplectic area   Maslov index

Note: L monotone $\Rightarrow$ $[\omega](\underset{\pi_2(X,L)}{u} \# \underset{\pi_2(X)}{v}) = [\omega](u) + [\omega](v)$

$$[\mu](u \# v) = [\mu](u) + 2c_1(X).[v]$$

$\Rightarrow [\omega](v) = 2\lambda c_1(X)[v]$ $\forall v \in \pi_2(X)$; ~~nearly~~ forces X monotone with $\tau = 2\lambda$.

For L monotone, N_L = minimal Maslov # of L
$= \{\min \mu(\alpha) \mid \alpha \in \pi_2(X, L), \mu(\alpha) > 0\}$

If L is orientable, $N_L \geq 2$, because any path of Lagr. lifts to the (oriented) double cover. From now on, assume $N_L \geq 2$.

Consequences of monotonicity:

1) Holomorphic spheres: if $u : S^2 \to X$ J-hol sphere, then the energy identity $\Rightarrow \int u^*\omega > 0 \Rightarrow c_1(X)[u] > 0$ by monotonicity.

Since any J-hol sphere is a branched cover of a simple (somewhere injective) J-hol sphere ([McDuff-Salamon]), monotonicity implies that Chern 1 spheres are simple, hence transversely cut out (by generic J)

Claim: Chern 1 spheres sweep out a "high codim subset of X".

Namely, when $c_1(A) = 1$,

$$\mathcal{M}_1(A) := \{u : (S^2, x_0) \to X \text{ J-hol} \mid u_*[S^2] = A\} / \text{reparam}$$

is a manifold (for generic J) of dimension

$$2n + 2\underbrace{c_1(A)}_{=1} + \underset{\text{marked pt}}{2} - \underset{\text{reparam}}{6} = 2n - 2$$

We have $ev : \mathcal{M}_1(A) \to X : u \mapsto u(x_0)$. The point is that codim ≥ 2.

We'll use this to argue that on $\partial($ index 2$)$, sphere bubbling does not occur.

[**Lemma**: on $\partial($ ind 2 $)$, sphere bubbling does not occur.

Proof: the only possible bubbling is 

, by count of dimension. □

2) We can count the holomorphic discs: [Lazzarini, Kwon-Oh]

(// McDuff's result) $\Rightarrow$ Maslov 2 hol discs with boundary on monotone L are simple, hence transversely cut out.

Given a class $\beta \in \pi_2(X, L)$ with $\mu(\beta) = 2$, let

$$\mathcal{R}^0(L;\beta, J) = \{u : (\mathbb{D}^2, \partial\mathbb{D}^2, \underset{\partial\mathbb{D}^2}{p}) \to (X, L, L) \text{ J-hol}\} / \text{Aut. of domain}.$$

By Lazzarini, Kwon-Oh, for generic J, this is a manifold of dim

$$n + \underbrace{\mu(\beta)}_{=2} + \underset{\text{marked pt}}{1} - \underset{\text{reparam}}{3} = n$$

There is a map $ev: \mathcal{R}^\circ(L;\beta) \to L : u \mapsto u(p)$.

Define $m_0(L) \in \Lambda$ by the following rule:

$$\sum_{\substack{\beta \in \pi_2(X,L)\\ \mu(\beta)=2}} T^{\omega(\beta)}\, ev_*[\mathcal{R}^\circ(L;\beta)] =: m_0(L)\cdot[L]$$

Rem: * we can show $\mathcal{R}^\circ(L;\beta)$ is compact: Gromov compactness + monotonicity $\Rightarrow$ no bubbles can happen


each has to have $\mu \geq 2$; impossible by monotonicity as $\int u^*\omega > 0$.

* Varying generic J results in a cobordant $\mathcal{R}^\circ(L;\beta)$: same argument as before: no bubble can appear.

More directly, can define $m_0(L) = \sum_{\substack{\beta\in\pi_2(X,L)\\ \mu(\beta)=2}} T^{\omega(\beta)}\, \# \mathcal{R}^\circ_q(L;\beta)$,

where $\mathcal{R}^\circ_q(L;\beta) := ev^{-1}(q) = \{u : (\mathbb{D}^2, \partial\mathbb{D}^2, p) \to (X, L, q)\}$.

Given a local system $\begin{matrix}\mathcal{E}\\ \downarrow \\ L\end{matrix}$ (write $(L,\mathcal{E})$), define

$$m_0(L,\mathcal{E}) = \sum_{\substack{\beta\in\pi_2(X,L)\\ \mu(\beta)=2}} T^{\omega(\beta)} \cdot \# \mathcal{R}^\circ_q(L;\beta)\cdot \underbrace{hol_{\mathcal{E}}(\partial\beta)}_{\in \Lambda^*}$$

ex: when L is monotone, can work over $\Lambda = \mathbb{C}$ with $T=1$. Indeed, by Gromov compactness, a fixed index disc has a priori determined energy by monotonicity.

ex.  Monotone $(S^1, \nabla^2$ rk 1 with holonomy $z \in \mathbb{C}^*)$

$m_0(S^1, \nabla^2) \approx z \pm \frac{1}{z}$ because these 2 discs go in opposite directions

If L_0, L_1 are monotone Lagrangians (potentially equipped with local systems):

[Proposition: $\mu^1 : CF^*(L_0,L_1) \circlearrowleft$ satisfies $(\mu^1)^2 = (m_0(L_0) - m_0(L_1))\cdot id$

$\Rightarrow$ if $L_0 = L_1$, then $(\mu^1)^2 = 0$

$\Rightarrow$ if $L_0 \neq L_1$, then $(\mu^1)^2 = 0 \iff m_0(L_0) = m_0(L_1)$.

Rem. $CF^*(L,L)$ can only be $\mathbb{Z}/2$-graded when L bounds a hol. disc of Maslov 2.

Sketch of proof: look at $\partial($  $)$.

- no sphere bubbles when X is monotone
- Fiber breaking , giving $(\mu^1)^2$.
- disc bubbles:  $\cup$ 

$\mu = 2 \Rightarrow$ ind $= 0$, ie cst strip. (by monotonicity)

So $p = q$, and we get discs through p, which is $m_0(L_1) \cdot \mathrm{id}(p)$ for the 1st picture, and similarly for the 2nd one. □

Hence, the unobstructed monotone Fukaya category, defined as

$$\hom_{\mathcal{F}^{uo}(X)}(L_0, L_1) = \begin{cases} CF^\circ(L_0, L_1) \circlearrowleft \mu^1 & \text{if } (\mu^1)^2 = 0 \\ 0 & \text{if } (\mu^1)^2 \neq 0 \end{cases},$$

decomposes as $\mathcal{F}^{uo}(X) := \bigoplus_{\lambda \in \Lambda} \mathcal{F}_\lambda(X)$, where $\mathcal{F}_\lambda(X) = \{L \mid m_0(L) = \lambda\}$ "L has charge λ".

In this case, each $\mathcal{F}_\lambda(X)$ is an A_∞-category over $\mathbb{C}, \mathbb{F}_p, \ldots, \Lambda$.

Proposition (unjustified today) If X is monotone, then $\mathcal{F}_\lambda \equiv 0$ (the "monotone Fukaya category with charge λ"), unless λ is an eigenvalue $c_1(X) \star - : QH^*(X) \circlearrowleft$

quantum multiplication, quantum cohomology: $H^*(X)$ as vector space, but product is deformed by J-hol spheres.

Rem: this is a shadow of a more general relationship between $QH^\circ(X)$ and $\mathcal{F}uk(X)$.

It's a little inconvenient / non-general for us to have the obstruction / charge $m_0(L) \in \Lambda = H^0(L)$. Instead, more generally, we want $\mu^0_L \in CF^*(L, L)$ as mentionned last class. We could try

not homology

$\#\{x \supset\}/\mathrm{aut} := \langle \mu^0_L, x \rangle$, but this has problems, as there is no automorphism-invariant choice of strip-like ends.

Use: various equivalent "Morse-Bott" type methods for computing the (curved) A_∞-algebra $(CF^*(L,L), \mu^\circ)$ by starting with $C^{sing}_\bullet(L)$ or $CM^\circ(L; f)$.

Morse-Bott Floer homology (FOOO):

Define $CF^*(L,L) := C_\bullet^{sing}(L)$ singular chains "in a suitable sense" (adically completed over Λ_{Nov}). For the μ^k's: instead of removing boundary marked points, equipping Σ with strip-like ends and counting $\{\ \ x\}$, put a boundary marked point z_i and require $u(z_i) \in$ image of some chain in L.

ex: μ^2: consider $\overline{R}^2(x;L,J,\beta) = \{J\text{-hol } z_0, z_1, z_2 \to (X,L)$, where the entire boundary including the z_i's map to $L\}$, and we have $ev_i: \overline{R}^2(\ldots) \to L : u \mapsto u(z_i)$, define

$$\mu^2(C_2, C_1) := \sum_{\beta \in \pi_2(X,L)} (ev_0)_* \left([\overline{R}^2(x;L,J,\beta)] \cap ev_1^* C_1 \cap ev_2^* C_2 \right).$$

⚠ have to define ev_i^*; this is a mess.

Constant discs contribute by the usual intersection product (⚠ not well-defined on singular chains). For μ^1, we don't count constant c; instead, send $c \mapsto \partial c$ (as singular chain) as "constant contribution".

[FOOO] in great generality, we can define a curved A_∞-structure on $C_\bullet(L)$ to $(H_\bullet(L), \mu^{k \geq 0}]$. So we can think of $\mu^0_L \in H_\bullet(L)$.

In the weakly unobstructed case: $\mu^0_L = \underset{\substack{\uparrow \\ \text{"charge"}}}{\lambda} [L] \in H_\bullet(L)$

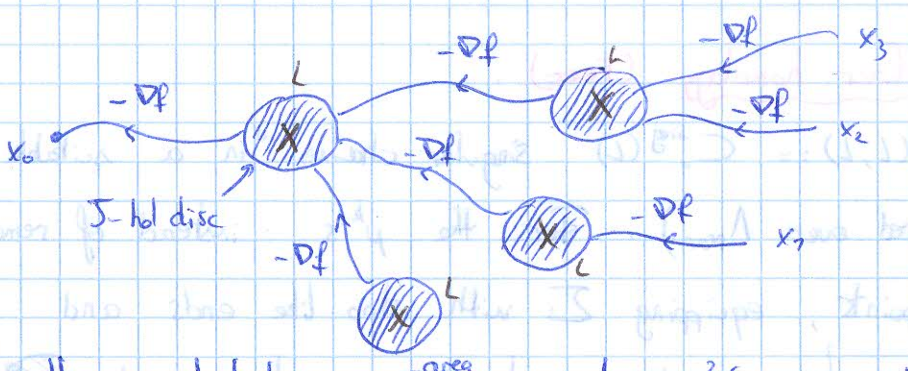
(always holds when L monotone).

Another approach: cluster homology

[Cornea - Lalonde]
[Biran - Cornea]
[Sheridan]

The idea is to count "clusters" as "pearly trees". Fix $f: L \to \mathbb{R}$ a Morse function, and let $CF^*(L,L) := CM^*(L,f) = \bigoplus_{x \in crit f} \Lambda\langle x \rangle$.

μ^k counts "clusters" of J-hol discs and Morse trajectories



this contributes $T^{\text{area}} x_0$ to $\mu^3(x_3, x_2, x_1)$.

Bubbling of discs is no longer a boundary in these moduli spaces
$\rightsquigarrow$ broken trajectories are boundaries.