And there is a fully faithful embedding $\mathcal{E} \hookrightarrow \mathcal{E}^{\Delta} \hookrightarrow \mathcal{E}^{\Delta,\pi}$
idempotent pre-Δ hull.

$\mathcal{E}$ always ⊛

11/05/16 **Definition**: an A_∞-category $\mathcal{E}$ is pretriangulated if

(a) • every closed morphism in $\mathcal{E}$ extends to an exact triangle

(b) • $\mathcal{E}$ is "closed under shifts": for every object X in $\mathcal{E}$, $\exists\, X[1]$ st

$\forall K$, $\begin{cases} hom^\circ_{\mathcal{E}}(X, K[1]) = hom^\circ(X,K)[1] \\ hom^\circ_{\mathcal{E}}(X[1], K)[1] = hom^\circ_{\mathcal{E}}(X,K) \end{cases}$ equality of chain complexes, compatible with A_∞-structures
⚠ signs

Rem: if (a), we can already define $X[1] := \mathrm{Cone}(X \xrightarrow{\circ} 0)$. Note that $0 = \mathrm{cone}(\Bbbk \xrightarrow{id} \Bbbk)$ is in the category.

⊛ (We always take $\mathcal{E}$ hom-unital).

But we also want $X[-1]$ to be define, hence (b).

Rem: $\mathcal{E}$ pretriangulated $\rightsquigarrow H^0\mathcal{E}$ triangulated in usual sense.

Proposition: any (hom-unital) A_∞-category $\mathcal{E}$ has a (essentially unique) pretriangulated hull, ie $\hat{\mathcal{E}}$ pretriangulated with $\mathcal{E} \to \hat{\mathcal{E}}$ such that if $\mathcal{D}$ pretriangulated with $\mathcal{E} \hookrightarrow \mathcal{D}$, then $\hat{\mathcal{E}} \hookrightarrow \mathcal{D}$.
In fact, the restriction $\mathrm{fun}(\hat{\mathcal{E}}, \mathcal{D}) \longrightarrow \mathrm{fun}(\mathcal{E}, \mathcal{D})$ is a quasi-equivalence.

Two methods of constructing this:

1) A_∞-modules over $\mathcal{E}$: construct a pretriangulated category $\mathrm{Mod}(\mathcal{E})$ with $\mathcal{E} \hookrightarrow \mathrm{Mod}(\mathcal{E})$, and take the closure of $\mathrm{im}(\mathcal{E})$ under finitely many cones and shifts.

2) Twisted complexes $\mathrm{Tw}\,\mathcal{E}$: construct directly from $\mathcal{E}$ an enlargment whose objects are "complex of objects"

ex. $\begin{tikzcd} X_0 \arrow[r, "c_{01}"] \arrow[rr, bend right, "c_{02}"'] & X_1 \arrow[r, "c_{12}"] & X_2 \end{tikzcd}$, $\mu^2(c_{12}, c_{01}) = \mu^1(c_{02})$: comp. is null homotopic instead of 0.

(Bondal - Kapranov, Kontsevich).

Let $\mathbb{k}$ be the ground field for $\mathcal{E}$. We will talk more about it.

Recall that given $\mathbb{k}$, we have a DG category $\mathrm{Chain}_{\mathbb{k}}$ ($\mathrm{Ch}_{\mathbb{k}}$):

* objects $C^\bullet \circlearrowright d_C$ ($C^\bullet$ graded vs, $\deg(d_C) = +1$) chain complex
* morphisms $\mathrm{Mor}(C_1, C_2) = \hom_{\mathrm{Vect}}(C_1, C_2)$, with d, inherits grading from C_1 and C_2. This differential is

(§) $d(F) := F \circ d_{C_1} - d_{C_2} \circ F$ ($\pm$ signs depending on $\deg F$), so closed morphisms are chain maps.

<u>Fact</u>: given A_∞-categories $\mathcal{E}$ and $\mathcal{D}$, we get $\text{nu-fun}(\mathcal{E}, \mathcal{D})$, category of (non-necessarily unital) A_∞-functors from $\mathcal{E}$ to $\mathcal{D}$. It is DG if $\mathcal{D}$ is.

<u>Definition</u>: the category $\mathrm{Mod}(\mathcal{E}) = \mathrm{Mod}\text{-}\mathcal{E}$ of (right) A_∞-~~functors~~ modules over $\mathcal{E}$ is $\text{nu-fun}(\mathcal{E}^{op}, \mathrm{Ch}_{\mathbb{k}})$. In fact, it is a DG category.

Let's explicitely spell this out.

<u>Definition</u>: a (right) A_∞-module P over $\mathcal{E}$ is the data of:

* for every $X \in \mathrm{ob}\,\mathcal{E}$, a chain complex $P(X) \circlearrowright \mu_P^{1|0}$, degree 1
* for every $(d+1)$-tuple of objects $X_0, \dots, X_d$, "multiplication maps"
  $\mu_P^{1|d} : P(X_d) \otimes \hom_{\mathcal{E}}(X_{d-1}, X_d) \otimes \dots \otimes \hom(X_0, X_1) \to P(X_0)$ of degree $1-d$.

Equivalently, $P^d : \hom(X_{d-1}, X_d) \otimes \dots \otimes \hom(X_0, X_1) \to \hom_{\mathrm{Ch}_{\mathbb{k}}}(P(X_d), P(X_0))$.

satisfying the A_∞-module equations (direct consequence of P A_∞-functor): for any k,

(*)
$$\sum_{i,j} \pm \mu_P^{1|k-j+1}(\underline{m}, x_k, \dots, x_{i+j+1}, \mu_{\mathcal{E}}^j(x_{i+j}, \dots, x_{i+1}), x_i, \dots, x_1) \quad (\mu_P^{1|0} \text{ allowed})$$
$$= \sum_i \pm \mu_P^{1|k-i}(\mu_P^{1|i}(\underline{m}, x_k, \dots, x_{k-i+1}), x_{k-i}, \dots, x_1)$$

<u>First few</u>:
* $(\mu_P^{1|0})^2(\underline{m}) = 0$
* $\pm \mu_P^{1|0}(\mu_P^{1|1}(\underline{m}, x)) = \pm \mu_P^{1|1}(\mu_P^{1|0}(\underline{m}), x) \pm \mu_P^{1|1}(\underline{m}, \mu_{\mathcal{E}}^1(x))$,

ie $\mu_P^{1|1}$ descends to cohomology: $H^\bullet(P(X)) \otimes \mathrm{Hom}(Y, X) \to H^\bullet(P(Y))$.

Rem: when $\mathcal{E}$ has one object X, $A := \hom_{\mathcal{E}}(X,X)$ A_∞-algebra.
A module over $\mathcal{E}$ $\rightsquigarrow$ "A_∞-module over A" = a graded vector space $M := P(X)$, with maps $\mu^{1|d}: M \otimes A^{\otimes d} \to M$ of deg $1-d$ $(d \geq 0)$ satisfying $(*)$.

Example: let K be any object in $\mathcal{E}$.
$\rightsquigarrow$ $Y_K^{(r)} :=$ "Yoneda module over $\mathcal{E}$" $\overset{\text{as functor}}{:=}$ $\hom_{\mathcal{E}}(-,K)$.

And
$$\mu_{Y_K}^{1|d}: \underbrace{Y_K(X_d)}_{\hom(X_d,K)} \otimes \hom(X_{d-1},X_d) \otimes \dots \otimes \hom(X_0,X_1) \xrightarrow{\mu_{\mathcal{E}}^{d+1}} \underbrace{Y_K(X_0)}_{\hom(X_0,K)}$$

If $\mathcal{E}$ has only one object X, then $A := \mathrm{End}(X)$, and we get A as an A_∞-module over itself.

What are morphisms in $\mathrm{Mod}(\mathcal{E})$?
(Answer: $\mathrm{Mor}(P_0,P_1) :=$ space of A_∞-pre-module morphisms $\mathcal{E} \to \mathcal{D}$).
A pre-morphism from P_0 to P_1 is the data of
- a linear map $F^{1|0}: P_0(X) \to P_1(X)$
- higher maps $F^{1|d}: P_0(X_d) \otimes \hom_{\mathcal{E}}(X_{d-1},X_d) \otimes \dots \otimes \hom_{\mathcal{E}}(X_0,X_1) \to P_1(X_0)$.

All together,
$$\mathrm{Mor}_{\mathrm{Mod}(\mathcal{E})}(P_0,P_1) := \prod_{\substack{X_0,\dots,X_d \in \mathrm{ob}\,\mathcal{E} \\ d\geq 0}} \hom_{\mathrm{vect}}\big(P_0(X_d) \otimes \hom(X_{d-1},X_d) \otimes \dots \otimes \hom(X_0,X_1),\ P_1(X_0)\big).$$

Shorthand: (borrowed from algebra case)
$$P_0 \otimes \mathcal{E}^{\otimes d} = \bigoplus_{X_0,\dots,X_d} P_0(X_d) \otimes \hom(X_{d-1},X_d) \otimes \dots \otimes \hom(X_0,X_1)$$
and
$$P_0 \otimes T\mathcal{E} := \bigoplus_{d \geq 0} P_0 \otimes \mathcal{E}^{\otimes d}.$$

Point: these $F^{1|d}$ do not satisfy any equations. Just like in (§).

Given a map $F := \oplus F^{1|d}: P_0 \otimes T\mathcal{E} \to P_1$ as before, there is a natural extension $\hat{F}: P_0 \otimes T\mathcal{E} \to P_1 \otimes T\mathcal{E}$,
$$\hat{F}(\underline{p}, x_d, \dots, x_1) := \sum \pm F(\underline{p}, x_d, \dots, x_{d-i+1}) \otimes x_{d-i} \otimes \dots \otimes x_1.$$

Similarly, given a A_∞-structure $\mu : T\mathcal{E} \to \mathcal{E}$, get a canonical

$\hat{\mu} : T\mathcal{E} \to T\mathcal{E}$: $\hat{\mu}(x_k, \dots, x_1) := \sum \pm x_k \otimes \dots \otimes \mu^j(x_{i+j}, \dots, x_{i+1}) \otimes x_i \otimes \dots \otimes x_1$.

<u>Composition</u>: given $G \in \hom_{Mod(\mathcal{E})}(P_1, P_2) :=$ "$\hom_{vect}(P_1 \otimes T\mathcal{E}, P_2)$"

$F \in \hom_{Mod(\mathcal{E})}(P_0, P_1)$,

define $G \circ F := G \circ \hat{F}$, namely

$G \circ F(\underline{m}, x_d, \dots, x_1) = \sum \pm G(F(\underline{m}, x_d, \dots, x_{d-i+1}), x_{d-i}, \dots, x_1)$.

and $\delta(F) := \mu_{P_1}^{1|0} \circ \hat{F} \mp F \circ \widehat{\mu_{P_0}^{1|0}} \mp F(\underline{m}, \hat{\mu}_{\mathcal{E}}(\dots))$

It is a DG category.

The association $K \rightsquigarrow Y_K^{(r)}$ extends to an A_∞-functor $Y_R : \mathcal{E} \to Mod(\mathcal{E})$.

ex: $Y_R^1 : \hom_{\mathcal{E}}(K, L) \to \hom_{Mod(\mathcal{E})}(Y_K, Y_L)$. Part of this data is, $\forall X$,

map $Y_K(X) \longrightarrow Y_L(X)$ for $\phi \in \hom_{\mathcal{E}}(K, L)$.

$\hom(X, K) \xrightarrow{\mu^2(-,\phi)} \hom(X, L)$

[Proposition (A_∞-Yoneda embedding) if $\mathcal{E}$ is hom-unital, Y_R is cohom. full and faithful.

[Corollary: any A_∞-category is quasi-equivalent to a DG category.

<u>Note</u>: $Mod(\mathcal{E})$ inherits the following operations from $Ch_{\Bbbk}$

(i) can take direct sum of modules: $P_0, P_1 \rightsquigarrow P_0 \oplus P_1(X) := P_0(X) \oplus P_1(X)$.

(ii) can tensor with a fixed chain complex or vector space:

V gr. v.s, P module $\rightsquigarrow$ $(V \otimes P)(X) = V \otimes (P(X))$.

(iii) can shift objects: $P[1](X) = P(X)[1]$.

(iv) mapping cones: recall that given a closed morphism $f : K^\bullet \to L^\bullet$ in $Ch_{\Bbbk}$,

get $Cone(f) \in ob(Ch_{\Bbbk})$

$K^\bullet[1] \oplus L^\bullet$, with $d_{Cone(f)} = \begin{pmatrix} d_{K^\bullet} & 0 \\ f & d_{L^\bullet} \end{pmatrix}$, filling into a triangle

$$\begin{tikzcd} K^\bullet \arrow[rr, "f"] & & L^\bullet \arrow[dl, "i"] \\ & \mathrm{Cone}(f) \arrow[ul, "{[1]}\, p"] & \end{tikzcd}$$

Rem: $\exists$ non trivial Massey product between p, i, f, namely $p \circ i = 0$ on chain level, but $i \circ f = \delta_{\mathrm{Cone}(f)}(\underbrace{pr_K}_{\alpha})$, and $\mathrm{Massey}(p, i, f) = p \circ \alpha = id_K$.

Similarly, given P_0, P_1 and $F : P_0 \to P_1$ closed morphism, have $\mathrm{Cone}(F) := P_0[1] \oplus P_1$, with

$$\mu^{1|d}_{\mathrm{Cone}(F)}\left(\begin{pmatrix} m_0 \\ m_1 \end{pmatrix}, \dots\right) = \begin{pmatrix} \mu^{1|d}_{P_0[1]}(m_0, \dots) \\ F^{1|d}(m_0, \dots) + \mu^{1|d}_{P_1}(m_1, \dots) \end{pmatrix}$$

Idempotents in $Ch_{\mathbb{k}}$, given $p \in \hom_{Ch_{\mathbb{k}}}(C^\bullet, C^\bullet)$ with $p^2 = p$ (idempotent), we can split off $C^\bullet = \underset{\circlearrowleft\, p\partial}{\mathrm{im}(p)} \oplus \underset{\circlearrowleft\, \partial - p\partial}{\dots}$

$\hookrightarrow$ chain map: $p\partial p \partial = \underbrace{pp\partial\partial}_{=0}$ as p closed

Similarly, given a A_∞-module P with "idempotent up to homotopy"
[Seidel] $\iff$ a morphism $[p] \in H^\bullet \hom_{\mathrm{Mod}(\mathcal{E})}(P, P)$ with $[p]^2 = [p]$,
we can split $P \simeq$ "$\mathrm{im}([p])$" $\oplus \dots$

Definition: the triangulated hull of $\mathcal{E}$, denoted $\widehat{\mathcal{E}}$ (or $Tw\,\mathcal{E}$), is the closure of $Y_R(\mathcal{E}) \subseteq \mathrm{Mod}(\mathcal{E})$ under finitely many shifts and mapping cones.

The split-closed triangulated hull of $\mathcal{E}$, denoted $Tw^\pi(\mathcal{E})$ or $\mathrm{Perf}(\mathcal{E})$ ("perfect modules"), is the closure of $Y(\mathcal{E})$ in $\mathrm{Mod}(\mathcal{E})$ under shifts, mapping cones and idempotent summands.

Both come equipped with embeddings (coh. full and faithful functors) $\mathcal{E} \hookrightarrow \widehat{\mathcal{E}} \hookrightarrow \mathrm{Perf}(\mathcal{E})$

If $\mathcal{A} \subseteq \mathcal{C}$ subcategory of $\mathcal{C}$, get $\hat{\mathcal{A}} \subseteq \hat{\mathcal{C}}$
and $\mathrm{Perf}(\mathcal{A}) \subseteq \mathrm{Perf}(\mathcal{C})$.

Definition: $D^{\pi}(\mathcal{C}) := H^0(\mathrm{Tw}^{\pi}\,\mathcal{C})$ idempotent Karoubi-completed derived category
$D(\mathcal{C}) := H^0(\mathrm{Tw}\,\mathcal{C})$ derived category.

Rem: $H^0(\mathrm{Tw}\,\mathcal{C}) \neq \mathrm{Tw}\,H^0\mathcal{C}$ (which is useless)

Rem. $A = \Bbbk[x]$ and $B = M_{n\times n}(\Bbbk[x])$ have same category of perfect modules.

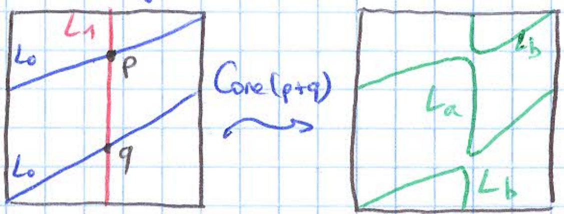
Definition: $\mathcal{A}$ $\left\{\begin{array}{l}\text{split-generates}\\ \text{generates}\end{array}\right.$ $\mathcal{C}$ if $\left\{\begin{array}{l}\mathrm{perf}\,\mathcal{A} \hookrightarrow \mathrm{perf}\,\mathcal{C}\\ \mathrm{Tw}\,\mathcal{A} \hookrightarrow \mathrm{Tw}\,\mathcal{C}\end{array}\right.$ is a quasi-equivalence
$\Leftrightarrow$ in $\left\{\begin{array}{l}\mathrm{perf}\,\mathcal{C}\\ \mathrm{Tw}\,\mathcal{C}\end{array}\right.$, every object of $\mathcal{C}$ is $\simeq$ to a summand of an iterated mapping cone of objects in $\mathcal{A}$.

<u>Nice fact:</u> in many cases, $\mathcal{F}uk(X)$ is split-generated by a subcategory with finitely many Lagrangians. So, $\mathcal{F}uk(X) \hookrightarrow A$-mod (or $\mathrm{perf}(A)$) for A some A_∞-algebra.

<u>Why</u> "split-generation"?

Philosophy: summand of objects $\rightsquigarrow$ connected components of Lagrangians.

<u>ex:</u> in T^2:



$L_a \amalg L_b$

$\rightsquigarrow$ L_0, L_1 "split-generate" L_a.