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16/05/16

Last time: A_∞-cat $\mathcal{E} \rightsquigarrow \mathrm{Tw}\,\mathcal{E} = \hat{\mathcal{E}}$ pre-Δ hull

$\mathrm{Tw}^\pi \mathcal{E} = \mathrm{Perf}(\mathcal{E})$ split-closed pre-Δ hull

A subcategory $\mathcal{A} \subseteq \mathcal{E}$ split-generates $\mathcal{E}$ if its inclusion induces a quasi-equivalence $\mathrm{Perf}(\mathcal{A}) \simeq \mathrm{Perf}(\mathcal{E})$.

Rem: the completion $\mathrm{Perf}(\mathcal{E})$ does not remember $\mathcal{E}$: of course $\mathcal{E} \hookrightarrow \mathrm{Perf}(\mathcal{E})$ is fully faithful, but we can't tell which objects come from $\mathcal{E}$.

Analogy:

$$\begin{array}{ccc}
\text{categories } \mathcal{E} & \rightsquigarrow & \text{subsets } X \subseteq \mathbb{R}^n \\
\downarrow \mathrm{Perf}(-) & & \downarrow \text{span} \\
\text{split-closed pre-}\Delta\ \mathcal{E} & \rightsquigarrow & \text{vector spaces} \subseteq \mathbb{R}^n
\end{array}$$

A bit more precisely:

$$\begin{array}{ccc}
\text{categories } \mathcal{E} & \rightsquigarrow & \text{subsets } X \subseteq \mathbb{R}^n \\
\downarrow \mathrm{Tw}(-) & & \downarrow \mathbb{Z}\text{-span} \\
\text{pre-}\Delta \text{ categories} & \rightsquigarrow & \text{lattices in } \mathbb{R}^n \\
\downarrow \mathrm{Tw}^\pi(-) & & \downarrow \\
\text{split-closed pre-}\Delta \text{ categories} & \rightsquigarrow & \text{"lattice of indecomposables"}
\end{array}$$

Example: given $\mathcal{A} \subseteq \mathcal{E}$, we can form an A_∞-quotient $\mathcal{E}/\mathcal{A}$ [Drinfeld: DG case, Lyubashenko-Manzyuk], and we have $\mathrm{Tw}^\pi \mathcal{E}/\mathcal{A} \simeq \dfrac{\mathrm{Tw}^\pi \mathcal{E}}{\mathrm{Tw}^\pi \mathcal{A}}$; moreover $\dfrac{\mathcal{E}}{\mathcal{A}} \simeq \{0\} \Leftrightarrow \mathcal{A}$ split-generates $\mathcal{E}$.

$\hookrightarrow$ or $\mathcal{E}/\mathcal{A} \simeq \mathcal{E}/\mathrm{Tw}^\pi \mathcal{A}$ when $\mathcal{E}$ is split-closed pre-Δ.

The quotient satisfies a universal property: there exists a functor $j: \mathcal{E} \to \mathcal{E}/\mathcal{A}$ which is "quasi"-initial among functors out of $\mathcal{E}$ sending $\mathcal{A} \to \{0\}$, meaning $\forall \mathcal{D}$:

$$\begin{tikzcd} & \mathcal{E}/\mathcal{A} \arrow[d, dashed] \\ \mathcal{E} \arrow[ur, "j"] \arrow[r, "f"] & \mathcal{D} \\ \mathcal{A} \arrow[u, hook, "i"] \arrow[r] & \{\text{a trivial subcat}\} \arrow[u, hook] \end{tikzcd}$$

$\circlearrowleft_h$ $\quad$ $\circlearrowleft_h$

A category is trivial if it is quasi-equivalent to $\{0\}$: one object X, with $End(X) = 0$.

Moreover, $m\text{-fun}(\mathcal{E}/\mathcal{A}, \mathcal{D}) \xrightarrow{\sim} m\text{-fun}_{\mathcal{A}\text{-triv}}(\mathcal{E}, \mathcal{D})$. We will do the explicit construction later.

Another type of quotient: the <u>localization</u> with respect to a class of morphisms: given $W \subseteq H^0\mathcal{E}$ (ie a subset of the morphism spaces $H^0 hom(X,Y)$ for each X,Y), we can define $\mathcal{E}[W^{-1}]$: the universal A_∞-cat (equipped with a map $j: \mathcal{E} \to \mathcal{E}[W^{-1}]$) in which the chain-level representatives of W become isomorphisms:

Univ prop: given $\mathcal{E} \to \mathcal{D}$ sending W to isos,

$$\begin{tikzcd} \mathcal{E} \arrow[r] \arrow[d, "j"'] \arrow[dr, phantom, "\circlearrowleft_h"] & \mathcal{D} \\ \mathcal{E}[W^{-1}] \arrow[ur, dashed, "\exists!"'] & \end{tikzcd}$$

If $\mathcal{E}$ is pre-triangulated, given W, define $Cones(W) \subseteq \mathcal{E}$ the full-subcategory whose objects are cones of the morphisms in W, and define $\mathcal{E}[W^{-1}] := \mathcal{E}/Cones(W)$ (cf cone of iso is trivial)

If $\mathcal{E}$ is not pre-Δ, define $\mathcal{E}[W^{-1}]$ to be the image of

$$\mathcal{E} \hookrightarrow Tw^{\pi}\mathcal{E} \xrightarrow{j} Tw^{\pi}\mathcal{E}/Cones(W)$$

Symplectic Landau-Ginzburg models:

These are singular fibrations which include Lefschetz fibrations, etc.

In physics / mirror symmetry, a LG model is a pair $(\check{X}, W)$, where
- $\check{X}$ is a non-compact Kähler manifold, $c_1(\check{X}) = 0$
- W is a holomorphic function on $\check{X}$.

HMS predicts: when X is not Calabi-Yau (ie $c_1(X) \neq 0$, eg X Fano), its "mirror" is a LG model $(\check{X}, W)$ with

$$D^\pi Fuk(X) \cong MF(\check{X}, W)$$
$$D^b Coh(X) \cong D^\pi Fuk(\check{X}, W)$$

We have seen a shadow of this when $X = \mathbb{P}^n$, with mirror $\left((\mathbb{C}^*)^n, W = z_1 + \dots + z_n + \frac{1}{z_1 \cdots z_n}\right)$. We saw that the category decomposes with respect to the charge, etc.

Goal: define a category $\mathcal{F}(\check{X}, W)$ associated to a LG model, and see how its structure / existence has implications for Fukaya categories of the fiber $\mathcal{F}(W^{-1}(p))$ (and perhaps $\mathcal{F}(\check{X})$).

Symplectic setup: (E^{2n+2}, W) where
- E is a non-compact symplectic manifold with technical hypotheses (exact, monotone, convex near ∞, ...)
- $W: E \to \mathbb{C}$ is a "symplectic fibration with singularities", meaning that away from $K_{cpct} \subseteq \mathbb{C}$, $W: E \setminus W^{-1}(K_{cpct}) \to \mathbb{C}$ is a genuine symplectic fibration ($\Rightarrow$ for each $p \in \mathbb{C} \setminus K_{cpct}$, $(M_p := W^{-1}(p), \omega|_{M_p})$ is a symplectic submanifold.

We assume that M_p is exact, monotone, ...

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Note: symplectic fibrations carry canonical connections: at a point $x \in M_p$, $T_x E = T_x M_p \oplus (T_x M_p)^{\perp_\omega}$

↳ canonical horizontal subspace.

$\Rightarrow$ We should get symplectic parallel transport maps (as long as the flow does not escape to ∞ if the fiber is non compact). We will assume that it is the case.

$\Rightarrow$ All fibers M_p are symplectomorphic.

Note: 

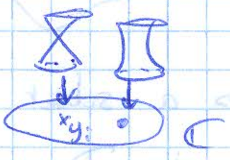
Get $\rho_\gamma : M \to M$ symplectic monodromy map.

Examples:

(1) Lefschetz fibrations: $\operatorname{crit} W = \coprod \{p_i\}$ ← points in $\mathbb{C}$

$E^{crit} = \coprod \{y_i\}$ ← crit pts living over p_i (generic case)

with a "holomorphic Morse" condition: at each y_j, a nbhd of (E, W) looks like a nbhd of 0 in $(\mathbb{C}^n, W = \sum_{i=1}^n z_i^2)$, and with various technical hypotheses.

Local pictures: 

(2) Lefschetz-Bott fibrations: $\operatorname{crit} W$ as before

$E^{crit} = \coprod \{Z_i\}$ ← sympl submfld of codim $2k_i$

← crit locus over p_i

with a "normal non-degeneracy" condition: a nbhd of $z \in Z_i$ looks like a nbhd of 0 in $(\mathbb{C}^n, \sum_{j=1}^{k_i} z_j^2)$.

(3) More general "stratified singularities". ex: $(\mathbb{C}^3, W = xyz)$. $\mathbb{C}^3_{crit} =$ 

There are many constructions of (1) and (2) in nature. For example, for $\bar{X}$ an algebraic compact variety and $\mathcal{L}$ an ample line bundle,

s_0, s_∞ sections of $\mathcal{L}$ with • $s_\infty^{-1}(0)$ smooth

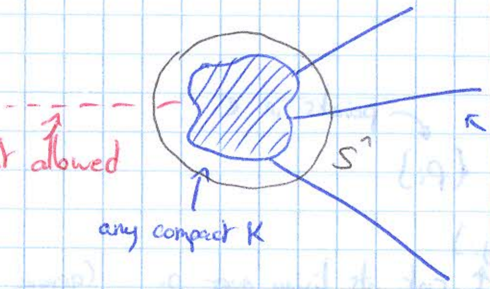
• $B := s_0^{-1}(0) \cap s_\infty^{-1}(0)$ smooth

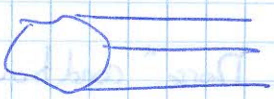
• s_0, s_∞ generic

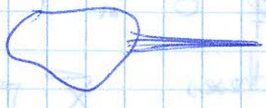
$$\Rightarrow \begin{array}{c} X := \bar{X} \setminus s_\infty^{-1}(0) \\ \big\downarrow {\scriptstyle s_0/s_\infty} \\ \mathbb{C} \end{array} \text{ is a Lefschetz fibration.}$$

Let (E, W) as before.

Definition: an admissible Lagrangian is a (possibly non-compact) Lagrangian $L^{n+1} \subseteq E^{2n+2}$ such that $W(L)$ is contained in:



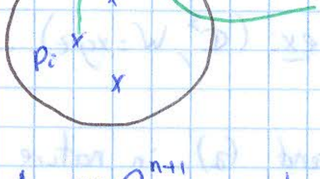
Other setups/choices: *  (straight lines // to $\mathbb{R}_+$)

or *  (asymptotic to a single line)

An admissible L induces a subset $D_L \subseteq (-\pi, \pi)$ of L's "directions near ∞". For L compact, $D_L = \emptyset$.

Key examples:

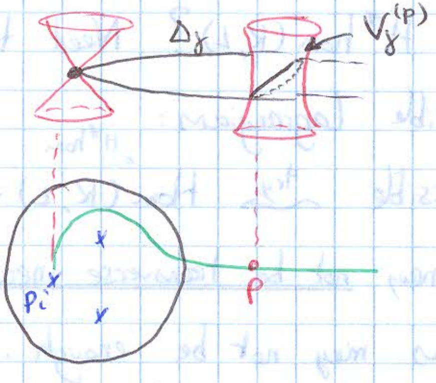
(1) (E, W) Lefschetz fibration, choose $\gamma : [0, \infty) \to \mathbb{C}$ with $\gamma(0) = p_i \in \mathrm{crit}\, W$, asymptotic to a radial ray, and st $\gamma(t) \in \mathbb{C} \setminus \mathrm{crit}\, W$ $\forall t \neq 0$:



Get Δ_γ a Lefschetz thimble := {points in $W^{-1}(\mathrm{im}\, \gamma)$ which parallel transport along γ to y_i (crit pt of W above p_i)}.

Note: $\Delta_\gamma \cong \mathbb{R}^{n+1}$, and $\Delta_\gamma|_{W^{-1}(\mathrm{im}\, \gamma|_{[0,3]})} \cong \mathbb{D}^{n+1}$.

Given $p \in \operatorname{im} \gamma$, $\Delta_\gamma \cap M_p =: V_\gamma^{(p)}$ "vanishing cycle" is a Lagrangian $S^n \subseteq M_p$.



(2) (E,W) Lefschetz-Bott: given γ as before and $L \subseteq Z_i^{2n+2-2k_i}$ (cnt sympl submfld over p_i) Lagrangian submfld, get

Δ_γ^L : generalized thimble

$:= \{$ points in $W^{-1}(\operatorname{im} \gamma)$ that parallel transport to L in Z_i along $\gamma \}$

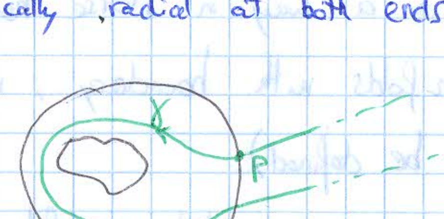
Check: Δ_γ^L is a $\mathbb{R}^{k_i+1}$-bundle over L

$\Delta_\gamma^L \cap W^{-1}(p)$ (for $p \in \operatorname{im}\gamma$) is a "generalized vanishing cycle", the total space of a S^{k_i}-fibration over L.

Canonical example: $\underset{\text{Lefschetz}}{(E,W)} \times \tilde{M} := \underset{\text{Lefschetz-Bott}}{(\tilde{M} \times E, W \circ \pi_E)}$,

$\Delta_\gamma \times \underset{\subseteq \tilde{M}}{\tilde{L}} \rightsquigarrow \Delta_\gamma^{\tilde{L}}$

(3) More generally, Lagrangians can have "multiple ends": given any (E,W), fix $\gamma : (-\infty, +\infty) \to \mathbb{C}$ going "around" $K_{cpct} \subseteq \mathbb{C}$ once, counterclockwise, asymptotically radial at both ends:



Fix $L \subseteq M_p$, where M_p is the reference fiber near ∞ going through γ near $t = -\infty$.

Get Lagrangian $U_\gamma^L := \{$ points in $W^{-1}(\operatorname{im}\gamma)$ which parallel transport to L along $\gamma \}$: "Orlov Lagrangian" [Orlov].

Towards defining $\mathcal{F}(E,w)$:

What should be $H^*\hom(K,L)$? Need to think of the Floer cohomology of admissible Lagrangians:

Given K, L admissible $\overset{try}{\rightsquigarrow}$ $\overset{H^*\hom}{\text{"}\mathrm{Hom}\text{"}}(K,L) := HF^*(\tilde{K}, L)$.

Problems: • (K,L) may not be transverse near ∞, so compactly supported perturbations may not be enough. This is especially visible in the setup where  (all same ray).

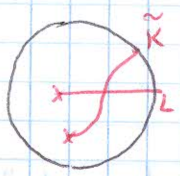
• discs may escape to ∞ $\Rightarrow$ failure of Gromov compactness.

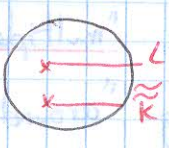
Tools for this: maximum principles. Decompose

$$\begin{array}{ccc} u: \mathbb{D}^2 & \longrightarrow & (E,J) \\ & \searrow^{\tilde{u}} & \downarrow w \\ & & \mathbb{C} \end{array}$$

If W is (J,j)-hol, then $\tilde{u}$ is j-hol near ∞. Then, use max. principle.

• lack of Hamiltonian isotopy invariance, when non-compact Ham. perturbations are allowed. This requires geometric thinking:

ex:  Prop: $HF^*(\tilde{K}, L) := HF^*(V_{\tilde{K}}, V_L)$, Floer hom in M_p.

But:  $HF^*(\tilde{\tilde{K}}, L) = 0$, even though $\tilde{\tilde{K}}$ Ham. isotopic to $\tilde{K}$.

This problem already occurs in a way in Morse homology for non-compact manifolds, or manifolds with boundary; namely $HM^*(f)$ not invariant (may not even be defined).

We can fix this by specifying an asymptotic profile of f near ∂M.

* "∇f points out": $HM^*(f)$ defined, and $\cong H^*(X)$,
* "∇f points in": $HD^*(f)$ defined, and $\cong H_*(X)$

which are not equal/PD when X has boundary.

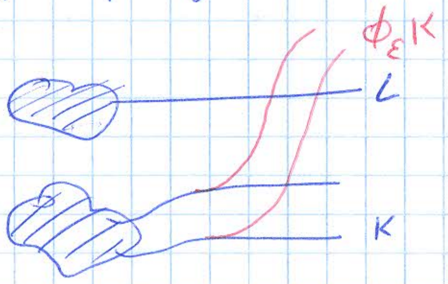
<u>Solution:</u> there is a distinguished choice of direction near ∞: "counterclockwise": ∂_θ near ∞, or wrt ω_{std} (a positive multiple) of the flow near ∞ of $h = r$.

Define $\mathrm{Hom}(K, L) := HF^*(\phi_\varepsilon \tilde{K}, L)$

↳ compactly supported perturbation

time-ε counterclockwise beat (lift to E of ε times this ham $h=r$ in $\mathbb{C}$).

ε sufficiently large so that all ends of K "above" those of L:



And we have a PSS isomorphism $\mathrm{Hom}(L, L) \cong H^*(L)$ (when L has one end at least).