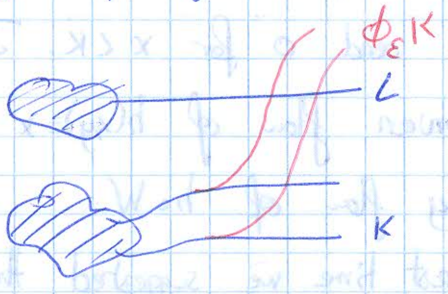
Solution: there is a distinguished choice of direction near ∞: "counterclockwise": ∂_θ near ∞, or wrt ω_{std} (a positive multiple) of the flow near ∞ of $h = r$.

Define $\mathrm{Hom}(K, L) := HF^*(\phi_\varepsilon \tilde{K}, L)$

↳ compactly supported perturbation

time-ε counterclockwise bent (lift to E of ε times this ham $h = r$ in $\mathbb{C}$).

ε sufficiently large so that all ends of K "above" those of L:



And we have a PSS isomorphism $\mathrm{Hom}(L, L) \cong H^*(L)$ (when L has one end at least).

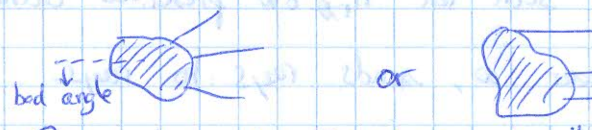
18/05/16 Last time: (E^{2n+2}, W) symplectic LG model,

* E non-cmpct symplectic (exact or monotone, ...)
* $W: E \to \mathbb{C}$ symplectic fibration away from $K_{cmpct} \subseteq \mathbb{C}$, with symplectic // transport maps (away from K_{cpct}).

$M := W^{-1}(p)$ general fiber: codim-2 symp. submanifold.

Towards $\mathcal{F}(E, W)$:

Objects: properly embedded $L \subseteq E$ (with brane structures) st $W(L)$ contained in



(can deform one into the other).

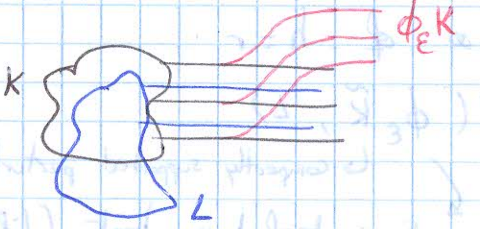
Given L, get $D_L \subseteq (-\pi, \pi)$ the angles, or $D_L \subseteq \mathbb{R}$ the asymptotic heights.

We say $D_K > D_L$ (or $K > L$) if $\theta_K > \theta_L$ $\forall \theta_K \in D_K$, $\forall \theta_L \in D_L$.

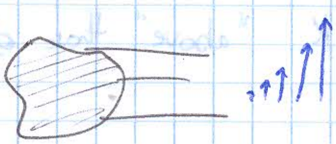
Last time, we argued what morphisms should look like in $H^\circ \mathcal{F}(E, W)$.

$H^\circ \hom_{\mathcal{F}}(K, L) := \mathrm{Hom}(K, L) = HF^\circ(\phi_\varepsilon K, L)$

where ϕ_ε is the time ε "admissible positive flow", "counterclockwise bent" (in the angular setup) which is large enough so that $\phi_\varepsilon K > L$:



What is an admissible flow? In the straight-lines setup, consider



the vector field on $\mathbb{C}$ which equals ∂_y for $x \gg 0$, and 0 for $x < K$. It is the Hamiltonian flow of $h(x,y) = x$ (when $x \gg 0$).

An admissible flow is any flow of $h \circ W$.

In the angular setup, last time we suggested that one can use flows of the form $h \circ W$ where $h = r$; X_h is ∂_θ on $\mathbb{C} \setminus \mathbb{D}^2$. Note that this flow is not admissible for a given flow for all times (of the lines can hit an angle π after being flown). We may not be able to find a flow $\phi_t K$ such that $\phi_t K$ is admissible $\forall t < \varepsilon$ with $\phi_\varepsilon K > L$:

Solution: allow flows which on $\mathbb{C} \setminus K_{cpt}$ have the form $h_{r,\theta} = \chi(\theta) h(r)$, where χ is a cutoff function equal to 0 at $-\pi$.

Lemma: Flowing by any time such an $h_{r,\theta} \circ W$ preserves admissibility.

Proof: check $h_{r,\theta}$ is radial: near ∞, sends rays to rays. ∎

For today: switch for horizontal ray framework for $\mathcal{F}(E, W)$.

Cohomological product: given L_0, L_1, L_2 admissible, define

$$[\mu^2] : \mathrm{Hom}(L_1, L_2) \otimes \mathrm{Hom}(L_0, L_1) \longrightarrow \mathrm{Hom}(L_0, L_2) :$$

choose $\varepsilon_1, \varepsilon_2$ so that $\phi_{\varepsilon_2+\varepsilon_1} L_0 > \phi_{\varepsilon_1} L_1 > L_2$; and count triangles:

$$HF^\circ(\phi_{\varepsilon_1} L_1, L_2) \otimes \underbrace{HF^\circ(\phi_{\varepsilon_1+\varepsilon_2} L_0, \phi_{\varepsilon_1} L_1)}_{\simeq\, HF^\circ(\phi_{\varepsilon_2} L_0, L_1)} \longrightarrow HF^\circ(\phi_{\varepsilon_1+\varepsilon_2} L_0, L_2)$$

Desired qualitative behavior of $\mathcal{F}(E,W)$:

* $L \simeq L'$ whenever L' is compactly Hamiltonian isotopic to L ;
* Invariance under admissible flows: meaning L_0 and $\phi_t L_0$ $\forall t$ should be quasi-isomorphic objects.

⚠ Why are we not doing this on chain level? For a pair (K,K), fix ε_K with $\phi_{\varepsilon_K} K > K$. Once and for all, set $\hom(K,K) := CF^\circ(\phi_{\varepsilon_K} K, K)$. Get

$$\cancel{\mu^2} : CF^\circ(\phi_{\varepsilon_K} K, K) \otimes CF^\circ(\phi_{\varepsilon_K} K, K)$$

$$\downarrow \mu^2$$

$$CF^\circ(\phi_{\varepsilon_K} K, K) \otimes CF^\circ(\phi^2_{\varepsilon_K} K, \phi_{\varepsilon_K} K)$$

$$\downarrow \mu^2$$

$$CF^\circ(\phi_\varepsilon K; K) \simeq CF^\circ(\phi^2_{\varepsilon_K} K; K)$$

But ...

Delicate to make a choice: the map $\leftarrow$ is not geometric.

~~Before~~

[Abouzaid-Seidel]: A_∞-construction of $\mathcal{F}(E,W)$

We use an auxiliary category $\mathcal{O}_W$

- $\mathrm{ob}\, \mathcal{O}_W :=$ admissible Lagrangian branes in $\mathcal{F}(E,W)$
- $$\hom_{\mathcal{O}_W}(K,L) := \begin{cases} CF^\circ(K,L) & \text{if } K > L \\ \mathbb{k}\langle e_L^+ \rangle & \text{if } K = L \ (\text{means } D_K = D_L \text{ and } \underline{K = L}) \\ 0 & \text{otherwise} \end{cases}$$

Notice that compact Lagrangians fit into this story: we can either say

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* if K compact, $D_K = \phi$, and $\phi <$ anything $\quad \mathbb{R}\setminus(-\pi,\pi) \supseteq$

or * consider for every K compact, all objects of the form (K, I)

This is a directed category, meaning $\mathrm{ob}\,\mathcal{O}_W$ is a poset with $\hom(K, L) = 0$ unless $K > L$.

This is easy to define as an A_∞-category: e_L^+ is a strict unit, and $\mu^k : \hom_{\mathcal{O}}(L_{k-1}, L_k) \otimes \dots \otimes \hom_{\mathcal{O}}(L_0, L_1) \to \hom_{\mathcal{O}}(L_0, L_k)$

$$= \begin{cases} 0 & \text{unless } L_0 > L_1 > \dots > L_k \\ \text{usual } \mu^k \text{ with } J\text{-hol disks} \end{cases}$$

Rem: * $L_i \pitchfork$ near ∞

* no need for Ham perturbations if L_i are pairwise transverse.

The A_∞-relations hold because "$L_0 > \dots > L_k$" is preserved under passing to linear subsequences of objects.

Notes: * ~~if L, L' differ by some compactly supported Ham. isotopy, then $L \simeq L'$ in $\mathcal{O}_W$.~~

* $L \not\simeq \phi_\varepsilon L$ because note

  * $\hom_{\mathcal{O}}(\phi_\varepsilon L, L) := CF^\circ(\phi_\varepsilon L, L) \overset{\neq 0}{\cong} C\Pi^\circ(L)$ (if ε small, L has one end)
  * $\hom_{\mathcal{O}}(L, \phi_\varepsilon L) = 0$

Definition: for every $L \in \mathrm{ob}\,\mathcal{O}$, there is a canonical element

$$q \in HF^\circ(\phi_\varepsilon L, L) := H^\circ(\hom_{\mathcal{O}}(\phi_\varepsilon L, L)) \quad \text{quasi-unit},$$

defined by $\sum \# \{\phi_\varepsilon L \ldots L\} \cdot x$, or by $1 \in C\Pi^\circ(L)$

The collection of all quasi-units gives a class of morphisms $Z \subseteq H^\circ \mathcal{O}_W$

Closure of Z; it satisfies $\mathcal{O}[Z^{-1}] = \mathcal{O}[\bar{Z}^{-1}]$.

Definition: [Abouzaid-Seidel] $\mathcal{F}(E, W) := \mathcal{O}[Z^{-1}]$.

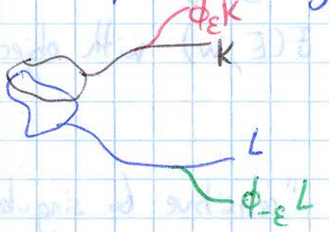
Note $L \sim L'$ in $\mathcal{F}(E,W)$: $\phi_{3\varepsilon} L' \to \phi_{2\varepsilon} L \to \phi_\varepsilon L' \to L$ w/ any pair of compositions in $\bar{Z}$.
(when differ by cpct. isotopy) have

By definition, $\mathcal{F}(E,W) \overset{\text{"}}{=} \mathcal{F}$ comes equipped with a functor $j: \mathcal{O} \to \mathcal{F}$ which is initial among functors sending Z to isomorphisms
$\Rightarrow$ can think of objects of $\mathcal{O}$ as objects of $\mathcal{F}$, and in $\mathcal{F}$, $\phi_\varepsilon L \simeq L$.

In order to compute the morphism spaces:

Proposition [Abouzaid-Seidel] "Correct position lemma" If $K > L$, then
$$j^1 : \underset{\substack{\| \\ CF^*(K,L)}}{\hom_{\mathcal{O}}(K,L)} \longrightarrow \hom_{\mathcal{F}}(K,L) \text{ is a quasi-isomorphism.}$$

Sketch: when $K > L$, multiplying by quasi-units on the right or left acts by cohomology isomorphisms.



ex: $\hom_{\mathcal{O}}(K,L) \otimes \hom_{\mathcal{O}}(\phi_\varepsilon K, K) \xrightarrow{\mu^2} \hom_{\mathcal{O}}(\phi_\varepsilon K, L)$

$\mu^2(-, q)$ is a quasi-isomorphism.

We don't create intersections. $\square$

(*)

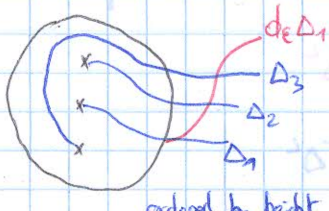
To compute products:

Proposition [Abouzaid-Seidel] "Correct position lemma part II"
If $L_0 > L_1 > \dots > L_k$, then for $x_i \in \hom(L_{i-1}, L_i)$, we have
$$j\, \mu^k(x_k, \dots, x_1) = \mu^k(j x_k, \dots, j x_1).$$
Meaning: we can compute the μ^k's in $\mathcal{O}$ for Lagrangians in "correct order".

(*) Can set up a model of $\mathcal{F}$ which is strictly unital (because $\mathcal{O}$ is) and such that $\hom_{\mathcal{O}}(K,L) \hookrightarrow \hom_{\mathcal{F}}(K,L)$ is an inclusion of chain complexes.

Rem: [Seidel]: $\mathcal{F}(E,W) = \mathcal{F}^{\mathbb{Z}/2}(E, W_{\text{doubled}})$:

Example: (E,W) is a Lefschetz fibration. Pick a basis of Lefschetz thimbles, let $V_i := \Delta_i \cap M$ be the vanishing cycle.
all paths disjoint



ordered by height

Note that (next page)

$$\mathrm{Hom}_{\mathcal{F}}(\Delta_i, \Delta_j) = \begin{cases} HF^*(V_i, V_j) & \text{if } i<j \\ 0 & \text{if } i>j \\ \Bbbk\langle [e_{\Delta_i}]\rangle & i=j \end{cases}$$

(intersection lies in fiber + max principle: discs stay in fiber)

<u>Note</u>: can also directly define $\mathcal{A} \subseteq \mathcal{F}(M)$ fiber, a <u>non-full</u> subcategory:

* $\mathrm{ob}\,\mathcal{A} = \{V_0, \dots, V_k\}$ corresponding to basis
* $$\hom(V_i, V_j) = \begin{cases} CF^*(V_i, V_j) & i<j \\ 0 & i>j \\ \Bbbk\langle e^+_{V_i}\rangle & i=j \end{cases}$$

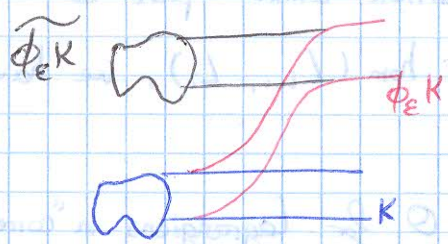
Applying the 2^{nd} Prop, + careful bookkeeping yields:

Theorem (Seidel) $\mathcal{A} \underset{\text{quasi-equiv}}{\simeq}$ the full subcat of $\mathcal{F}(E, W)$ with objects $\{\Delta_0, \dots, \Delta_k\}$.

$\mathcal{F}(E, W)$ is an invariant of $W: E \to \mathbb{C}$ "sensitive to singularities of W".

ex: Consider $W = \pi_{\mathbb{C}} : M \times \mathbb{C} \to \mathbb{C}$; it has no singularity.

Claim: $\mathcal{F}(M \times \mathbb{C}, \pi_{\mathbb{C}}) \simeq 0$. Indeed, we have isomorphisms of objects:



$$K \simeq \phi_\varepsilon K \underset{\substack{\text{compact} \\ \text{isotopy}}}{\simeq} \widetilde{\phi_\varepsilon K},$$

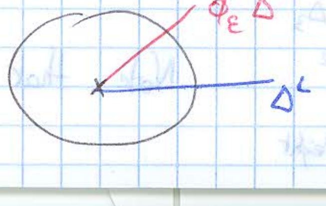
get $\hom_{\mathcal{O}}(\widetilde{\phi_\varepsilon K}, K) = \hom_{\mathcal{F}}(\widetilde{\phi_\varepsilon K}, K)$.
$\quad\quad\quad \overset{\shortparallel}{0}$

ex: $\mathcal{F}(\mathbb{C}^n, \sum_{i=1}^n z_i^2) \ni \Delta$

$\mathrm{Hom}(\Delta, \Delta) = \Bbbk$.

Proposition: Δ split-generates $\mathcal{F}(\mathbb{C}^n, \sum_{i=1}^n z_i^2)$, so $Tw^\pi \mathcal{F}(\mathbb{C}^n, \sum z_i^2) \simeq \mathrm{Perf}(\Bbbk) \simeq Ch^{fin}_{\Bbbk}$, chain complexes with finite cohomology.

ex: (E, W) Lefschetz-Bott fibration with one critical component Z (ex: $M \times \mathbb{C}^n$, $W = \sum_{i=1}^n z_i^2$).



Draw γ: for every $L \subseteq Z$, get a generalized thimble Δ^L.
Note that it seems that

$$\mathrm{Hom}(\Delta^L, \Delta^K) := HF^\circ(\phi_\varepsilon \Delta^L, \Delta^K) \cong HF^\circ(L, K) \quad \text{in } Z.$$

Theorem [Abouzaid-Auroux-Katzarkov] This is true (with signs) (for Lagrangian branes when $\nu(Z)$ (normal bundle) is spin.

Proposition. [AAK, A-Ganatra] there is a fully faithful A_∞-embedding

$$\mathcal{F}(Z) \xhookrightarrow[\Delta^{(\cdot)}]{} \mathcal{F}(E, W)$$

Theorem: $\Delta^{(\cdot)}$ (any split-generating collection in Z in sense of satisfying "Abouzaid's generation criterion") split-generates.

↳ strictly stronger than split-generation.

Next time:

$$\phi_{2\pi} \circlearrowright \mathcal{F}(E,W) \underset{\cup}{\overset{\cap}{\rightleftarrows}} \mathcal{F}(M) \circlearrowleft \mu \text{ monodromy map}$$

↑ general fiber.