Draw γ: for every $L \subseteq Z$, get a generalized thimble Δ^L.
Note that it seems that

$$\mathrm{Hom}(\Delta^L, \Delta^K) := HF^\bullet(\phi_\varepsilon \Delta^L, \Delta^K) \cong \overset{\text{in } Z}{HF^\bullet(L,K)}$$

Theorem [Abouzaid-Auroux-Katzarkov] This is true (with signs) (for Lagrangian branes when $\nu(Z)$ (normal bundle) is spin.

Proposition. [AAK, A-Ganatra] there is a fully faithful A_∞-embedding

$$\mathcal{F}(Z) \xhookrightarrow[\Delta^{(\cdot)}]{} \mathcal{F}(E,W)$$

Theorem: $\Delta^{(\cdot)}$ (any split-generating collection in Z in sense of satisfying "Abouzaid's generation criterion") split-generates.

↳ strictly stronger than split-generation.

Next time:

$$\begin{tikzcd} \mathcal{F}(E,W) \arrow[loop left, "\phi_{2\pi}"] \arrow[r, shift left, "\cap"] & \mathcal{F}(M) \arrow[l, shift left, "\cup"] \arrow[loop right, "\mu"] \end{tikzcd}$$

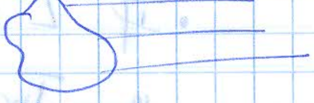
μ monodromy map

M: ↑ general fiber

23/05/16: Last time: (E,W) symplectic LG-model, e.g. a symplectic fibration with singularities. Take p near ∞, then the "general fiber" $M := W^{-1}(p)$ is a symplectic submanifold.

↳ Get $\mathcal{F}(E,W)$ and $\mathcal{F}(M)$.

Today: How are these two related?



Recall: ob $\mathcal{F}(E,W)$ = W-admissible Lagrangians, with $D_L \subseteq \mathbb{R}$ the set of heights. On cohomology,

$$\mathrm{Hom}_{\mathcal{F}(E,W)}(K,L) = HF^\bullet(\phi_{\varepsilon_K} K, L)$$

↑ sufficiently large "admissible positive flow"

The cohomology of the other definition (with oriented stuff) always coincides with this.

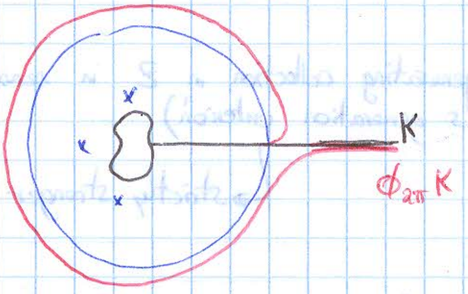
Idea: can define functors, which are adjoint:

$$\begin{tikzcd} \mathcal{F}(E,W) \arrow[loop left, "\phi_{2\pi}"] \arrow[r, shift left, "\cap"] & \mathcal{F}(W) \arrow[l, shift left, "U_{ccw}"] \arrow[l, bend left=60, "U_{cw}"] \arrow[loop right, "\mu"] \end{tikzcd}$$

$\cap$ is "cap", some intersection.

U_{ccw} and U_{cw} are the Orlov functors, counterclockwise & clockwise

- * μ is the global monodromy, induced by // transport around large loop.
- * $\phi_{2\pi}$ is "once wrapping" counterclockwise":



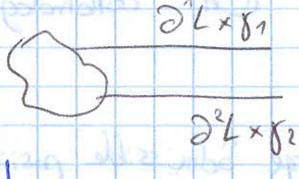
In general, there is a $\phi_{2\pi s}$ $\forall s \in \mathbb{Z}$.

Proposition: [Kontsevich, Seidel] $\phi_{-2\pi}$ is the "Serre functor", up to degree shift.

- * $\cap : \mathcal{F}(E,W) \to \mathcal{F}(M)$ is "intersection with a fiber"
  [Abouzaid-Seidel, Abouzaid Ganatra].
  - If L has 1 horizontal end, $\cap : L \mapsto \partial L := L \cap W^{-1}(p)$

$\longmapsto V = \partial\Delta$ vanishing cycle

  - If L has multiple ends, we should think of $\cap$ as landing in $\mathrm{Tw}\,\mathcal{F}(M)$, $\mathrm{Perf}\,\mathcal{F}(M)$, ...



$\longmapsto \mathrm{Cone}(\partial^1 L \xrightarrow{\alpha} \partial^2 L)$,

where α is some morphism coming from count of discs in the total space

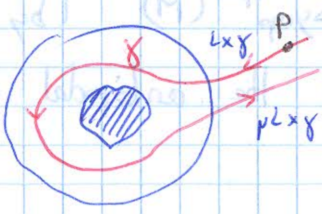
- No end: L maps to $0 \in \mathrm{Perf}$.

[cf Biran-Cornea's work on lagr. cobordisms]

* $U_{ccw} : \mathcal{F}(M) \longrightarrow \mathcal{F}(E, w)$ "Orlov functor": $L \longmapsto U_L^{\gamma}$.

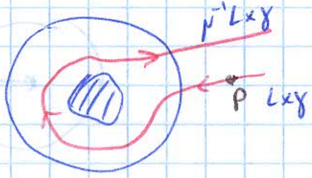
$U_L^{\gamma} := \{$ pts in $w^{-1}(\text{im}\, \gamma)$ which // transport to L along $\gamma \}$

U_{ccw}



Note: $U_{cw} = U_{ccw} \circ \mu^{-1}$, so let's not talk about U_{cw}.

U_{cw}:

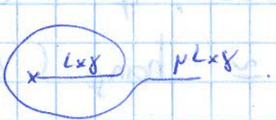


These structures are related; for instance

Proposition: [Abouzaid-Ganatra] up to a degree shift (by 2 ?),

(1) $\cup \mu \simeq \phi_{2\pi} \cup$

(2) $\cap \phi_{2\pi} \simeq \mu \cap$:



Theorem: [Abouzaid-Ganatra] there are exact triangles of functors.

(a) In $\mathcal{F}(E, W)$:

$$\begin{tikzcd} id \arrow[rr] & & \phi_{2\pi} \arrow[dl] \\ & \cup\cap \arrow[ul] & \end{tikzcd}$$

, meaning $\forall L \in \text{ob}\, \mathcal{F}(E, w)$, there is a triangle

$$\begin{tikzcd} L \arrow[rr] & & \phi_{2\pi} L \arrow[dl] \\ & \cup\cap L \arrow[ul] & \end{tikzcd}$$

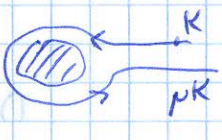
(b) In $\mathcal{F}(M)$:

$$\begin{tikzcd} id \arrow[rr] & & \mu \arrow[dl] \\ & \cap\cup \arrow[ul] & \end{tikzcd}$$

, meaning $\forall K \in \text{ob}\, \mathcal{F}(M)$, there is a triangle

$$\begin{tikzcd} K \arrow[rr] & & \mu K \arrow[dl] \\ & \cap\cup K \arrow[ul] & \end{tikzcd}$$

Note by definition: $\cap\cup K \simeq$ some $\text{Cone}(K \to \mu K)$:



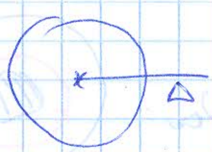
Example application: (LES of a Dehn twist)

Given $L \subseteq M$ a (parametrized) Lagrangian sphere in M, there is a Lefschetz fibration (E, w) with one critical point, with fiber M and with vanishing cycle $\simeq L$.

The Dehn twist τ_L can be defined to be (up to Hamiltonian isotopy) the monodromy $\mu : M \to M$ associated to (E, w).

Or, define τ_L directly as an element of $\mathrm{Symp}^{cpct}(\overset{T^*S^n}{\overset{\wr\|}{T^*L}})$, and hence as an element of $\mathrm{Symp}^{cpct}(M)$ (by existence of Weinstein neighbourhoods). τ_L is the antipodal map on L, and the identity far from L.

ex:



We know for this (E, w):

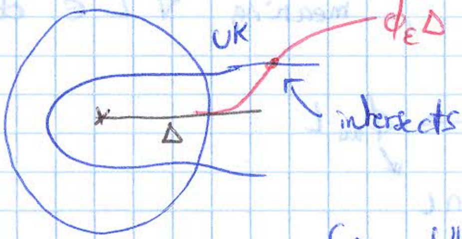
Theorem: [Seidel, Abouzaid-Ganatra] the thimble Δ generates $\mathcal{F}(E,w)$, and moreover $\mathrm{End}_{\mathcal{F}}(\Delta) \simeq \Bbbk$.

As a consequence, if $K \in \mathrm{ob}\,\mathcal{F}(M)$ is arbitrary, then

$$\underset{\substack{\downarrow \\ \text{object of } \mathcal{F}(E,w)}}{\cup K} \simeq \hom_{\mathcal{F}}(\Delta, \cup K) \otimes \Delta$$

considered as an element of $\mathrm{Perf}(\mathcal{F}(E,w))$

↳ projection of $\cup K$ onto the category split-generated by Δ.



intersects exactly at $\overset{\partial\Delta}{\overset{\|}{L}} \cap K$

↳ $\cup K \simeq CF^*(L,K) \otimes \Delta$.

Hence, the tautological exact triangle
$$\begin{tikzcd} K \arrow[r] & \tau_L K \arrow[dl] \\ \cap \cup K \arrow[u, "{[1]}"] & \end{tikzcd}$$
becomes

$$\begin{tikzcd} K \arrow[r] & \tau_L K \arrow[dl] \\ \cap(CF^*(L,K) \otimes \Delta) \arrow[u, "{[1]}"] & \end{tikzcd}$$
, which is
$$\begin{tikzcd} K \arrow[r] & \tau_L K \arrow[d] \\ & CF^*(L,K) \otimes L \arrow[ul] \end{tikzcd}$$

which reproves the LES of a Dehn twist [Seidel].

Need to do (for instance): build a functor

$$\cap : \mathcal{F}(E,w) \longrightarrow \mathcal{F}(M).$$

First, we will give a different construction of $\mathcal{F}(M)$, which is more manifestly compatible with Abouzaid-Seidel's $\mathcal{F}(E, W)$ ([Seidel], essentially using [Abouzaid-Seidel]).

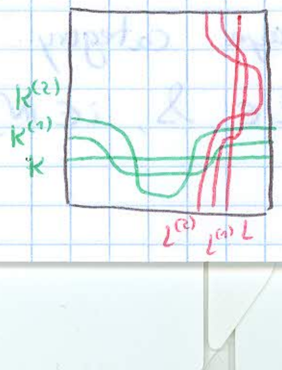
Let X/M be a symplectic manifold.

- Fix a set of objects $\mathcal{S} = \text{ob}\, \mathcal{F}(X)$ of Lagrangian branes.
  For technical reasons, assume $\mathcal{S}$ is countable.
- For each $L \in \mathcal{S}$ and $i \in \mathbb{N}$, pick a Ham. perturbation $L^{(i)}$, satisfying - any finite subset $(L_{i_0}^{(k_0)}, \dots, L_{i_d}^{(k_d)})$ with all k_i distinct are in general position (pairwise transverse).
  (only really need this for $k_0 > \dots > k_d$).
  (easy to do by induction, by countability)

Define a directed A_∞-category $\mathcal{O}_M$ as follows:

- $\text{ob}\, \mathcal{O}_M := \{ L^{(k)} \mid L \in \mathcal{S},\ k \in \mathbb{N} \}$
  (or rather pairs (L, k), so that $\mathcal{O}_M$ remembers the integer chosen)
- $$\hom_{\mathcal{O}_M} (L_0^{(k_0)}, L_1^{(k_1)}) = \begin{cases} CF^*(L_0^{(k_0)}, L_1^{(k_1)}) & \text{if } k_0 > k_1 \\ \mathbb{k} \langle e^+_{L_0} \rangle & \text{if } k_0 = k_1 \text{ and } L_0 = L_1 . \\ 0 & \text{otherwise} \end{cases}$$
- the A_∞-structure maps only non-zero for $k_0 > k_1 > \dots > k_d$ (+ case of equality : strict unitality). Moreover, in this case, $\{ L_i^{(k_i)} \}_{i=0}^{d}$ are pairwise transverse, so in the count of discs, we can set the Hamiltonian term to be equal to zero. So we are genuinely counting J-hol discs, albeit with potentially domain-dependent J in higher dimensions.

ex. $T^2 \setminus pt$:



In the Riemann surface case, $\mathcal{O}_M$ is essentially combinatorial, though there are infinitely many objects.

Define $Z_M \subseteq H^0(\mathcal{O}_M)$ the "quasi-units":

$q \in \hom^0_{\mathcal{O}_M}(K^{(s)}, K^{(t)})$ whenever $s > t$, elements

$\shortparallel$

$CF^0(K^{(s)}, K^{(t)})$ defined by (eg) continuation maps:

$q = \sum_x \#\Big($  $\Big) \cdot x$

These are the elements that should induce isomorphisms $K^{(s)} \simeq K^{(t)}$ in $\mathcal{F}(M)$.

$\hookrightarrow$ Define $\mathcal{F}_M^{loc} := \mathcal{O}_M[Z_M^{-1}]$. Have $j : \mathcal{O}_M \to \mathcal{F}_M^{loc}$

Proposition: there is a quasi-equivalence $\mathcal{F}_M^{loc} \simeq \mathcal{F}(M)$, where $\mathcal{F}(M)$ is the Fukaya category with objects $\mathcal{S}$.

Rem: in the exact case, if K, K' ham. isotopic in $\mathcal{F}(M)$,

$$\begin{tikzcd} \hom(K,K') \arrow[r, phantom, "\simeq"] & \hom(K,K) \arrow[r, phantom, "\simeq"] & H^*(K) \\ q \arrow[u, phantom, "\in" rotate=90] \arrow[rr, leftrightarrow] & & 1 \arrow[u, phantom, "\in" rotate=90] \end{tikzcd}$$

$\mu^2(-, q) : \hom(L, K) \xrightarrow{\sim} \hom(L, K')$

Lemma (analogue of "correct position lemma") if $s > t$, then
$j^1 : \hom_{\mathcal{O}_M}(K^{(s)}, L^{(t)}) \xrightarrow{\sim} \hom_{\mathcal{F}_M^{loc}}(K^{(s)}, L^{(t)})$.

In particular, $\hom_{\mathcal{F}_M^{loc}}(K^{(s)}, K^{(t)}) \simeq \hom_{\mathcal{F}_M^{loc}}(K^{(s+1)}, K^{(s)})$

$\wr$ Lemma

$\hom_{\mathcal{O}_M}(K^{(s+1)}, K^{(s)}) \simeq CF^*(K^{(s+1)}, K^{(s)})$

Rem. a $K^{(s)}$ in $\mathcal{F}(M)$ corresp. to $\cdots \to K^{(s+1)} \to K^{(s)} \to K^{(s-1)} \to \cdots$ in $\mathcal{O}_M$.

Sketch of proof of Prop.

Define $\mathcal{F}_M^{big}$ a version of Fukaya category defined as usual, with objects $\{L^{(i)} \mid L \in \mathcal{S},\ i \in \mathbb{N}\} \cup \{L \mid L \in \mathcal{S}\}$.

* We can choose the perturbations involved in μ^d to agree on the nose : to agree with $\mathcal{F}(M)$ when all objects are in $\{L \mid L \in \mathcal{S}\}$.

$\hookrightarrow$ $\mathcal{F}(M) \overset{\sim}{\hookrightarrow} \mathcal{F}^{big}(M)$. Note that this is a quasi-equivalence : each $L^{(i)} \simeq L$ for $L \in \mathcal{S}$.

* We can also choose perturbations such that for $(L_0^{(k_0)}, \dots, L_d^{(k_d)})$ with $k_0 > \dots > k_d$, agree with choices made in $\mathcal{O}_M$. This essentially gives $\exists\, F : \mathcal{O}_M \to \mathcal{F}_M^{big}$ a strict functor (ie F^d are zero for $d > 1$). We have to worry ~~ab~~ a bit about strict units, but let's not.

* Note that in $\mathcal{F}^{big}(M)$, morphisms in Z are sent to isomorphisms. By universal property,

$$\begin{tikzcd}
\mathcal{O}_M \arrow[r, "F"] \arrow[d, "j"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathcal{F}_M^{big} \\
\mathcal{F}_M^{loc} \arrow[ur, dashed, "\exists \overline{F}"'] &
\end{tikzcd}$$

* All we need to check is that $\overline{F}$ is cohom. fully faithful, because then

$$\begin{tikzcd}
\mathcal{F}_M^{loc} \arrow[rr, "\overline{F}", "\sim"'] \arrow[dr, dashed, "\text{by inverting q-equiv}"'] & & \mathcal{F}_M^{big} \\
& \mathcal{F}(M) \arrow[ur, "\sim"'] &
\end{tikzcd}$$

* Cohom fully faithfulness of $\overline{F}$: for $K^{(s)}, L^{(t)} \in \operatorname{ob} \mathcal{F}_M^{loc}$

$$\begin{tikzcd}
& \hom_{\mathcal{F}_M^{loc}}(K^{(s)}, L^{(t)}) \arrow[r, phantom, "\simeq"] \arrow[dl, "\overline{F}"'] & \hom_{\mathcal{F}_M^{loc}}(K^{(s+N)}, L^{(t)}) \\
\hom_{\mathcal{F}_M^{big}}(K^{(s)}, L^{(t)}) \arrow[d, phantom, "\wr"] & \text{by correct pos. lemma} \arrow[r] & \hom_{\mathcal{O}_M}(K^{(s+N)}, L^{(t)}) \arrow[u, "j", "\wr"'] \arrow[d, equal] \\
\hom_{\mathcal{F}_M^{big}}(K^{(s+N)}, L^{(t)}) & & CF^*(K^{(s+N)}, L^{(t)}) \arrow[ll, "\cong", "\overline{F}"']
\end{tikzcd}$$

because $K^{(s)} \simeq K^{(s+N)}$, with N big enough so that $s + N > t$

(just chain-level bijection)