25/05/16: Rem: about $UK \simeq \hom_{\mathcal{F}}(\Delta, UK) \otimes \Delta$. For $\mathcal{A} \overset{i}{\subseteq} \mathcal{E}$ such that $\mathcal{A}$ generates $\mathcal{E}$, then any object $K \in ob\,\mathcal{E}$ is isomorphic to $i_* i^* K$, for adjoint functors $Mod(\mathcal{A}) \underset{i^*}{\overset{i_*}{\rightleftarrows}} Mod(\mathcal{E})$.

* Analogy: $(Cat, \mathcal{E}^{op} \times \mathcal{E} \xrightarrow{\hom(-,-)} Ch_{\Bbbk}) \longleftrightarrow (Vect, \text{inner product } V \times V \xrightarrow{\langle -,- \rangle} \Bbbk)$.

  $W \overset{i}{\subseteq} V$ map of v.s. with inner product. We have a projection $i_* i^* : V \to V$. And $W = V \iff i_* i^* = id$.

* Given an object $K \in ob\,\mathcal{E}$, there is a canonical module $i_* i^* K \in Mod(\mathcal{E})$,

(β) $$i_* i^* K(X) := \hom_{\mathcal{E}}\Big(X, \underbrace{\bigoplus_{A_0, \dots, A_s \in ob\,\mathcal{A}} \hom_{\mathcal{E}}(A_s, K) \otimes \hom_{\mathcal{A}}(A_{s-1}, A_s) \otimes \dots \otimes \hom_{\mathcal{A}}(A_0, A_1) \otimes A_0}_{\text{"} i_* i^* K \text{"}}\Big)$$

rather Y_{A_0}

<u>Proposition:</u> $\mathcal{A}$ split-generates $\mathcal{E}$ $\iff i_* i^* K \simeq K$

$\iff K$ is a summand of $i_* i^* K$

$\iff 1 = 0$ in $\hom_{\mathcal{E}/\mathcal{A}}(K,K) := Cone\big(\hom_{\mathcal{E}}(K, i_* i^* K) \xrightarrow{\mu} \hom_{\mathcal{E}}(K,K)\big)$

Now, in the setting where one thimble Δ generates, let $\mathcal{A}$ be the subcategory with one thimble. There is a "minimal model" $\mathcal{A} \overset{q\text{-equiv}}{\simeq} \mathcal{A}_{min}$ with $ob\,\mathcal{A}_{min} = \{\Delta\}$, $End_{\mathcal{A}_{min}}(\Delta) = \Bbbk\langle e_\Delta^+ \rangle$.

<u>Lemma:</u> when $\mathcal{A}$ is strictly unital and <u>augmented</u> (ie $\hom_{\mathcal{A}}(X,X) \xrightarrow{f_X} \Bbbk$, $e_X^+ \mapsto 1$), define the <u>augmentation ideal</u> by

$$\hom_{\overline{\mathcal{A}}}(X,Y) = \begin{cases} \hom_{\mathcal{A}}(X,Y) & \text{if } X \neq Y \\ \ker(f_X) & \text{if } X = Y \end{cases}$$

(so, it is everything but the e_X^+'s). The lemma is that there is a complex, quasi-isomorphic to (β), that consists in (β) with all the $\mathcal{A}$'s replaced by $\overline{\mathcal{A}}$.

$\hookrightarrow$ For our $\mathcal{A}_{min} \subseteq \mathcal{F}(E,W)$ given by the thimble Δ, we get

$i_* i^* K = \hom_{\mathcal{F}}(\Delta, K) \otimes \Delta$, and no other terms.

Last time: (E,W) LG model, M fiber.

We discussed functors (at least cohomologically)

$$\phi_{2\pi} \circlearrowright \mathcal{F}(E,W) \underset{\cup}{\overset{\cap}{\rightleftarrows}} \mathcal{F}(M) \circlearrowleft \mu$$, with 2 exact triangles:

- In $\mathcal{F}(E,W)$:
$$\begin{tikzcd} id \arrow[r] & \phi_{2\pi} \arrow[dl] \\ \cup\cap \arrow[u] & \end{tikzcd}$$
- In $\mathcal{F}(M)$:
$$\begin{tikzcd} id \arrow[r] & \mu \arrow[dl] \\ \cap\cup \arrow[u] & \end{tikzcd}$$

Let's understand the first one more.

More on $\phi_{2\pi}$: "once wrapping". There is also $\phi_{2\pi k}$, $k \in \mathbb{Z}$.
We saw that, at least cohomologically, $\phi_{-2\pi}$ is the Serre functor, up to a shift by n.

Definition: $\mathcal{C}$ category. A Serre functor (shifted by n) is an automorphism $S: \mathcal{C} \to \mathcal{C}$ such that $\forall X, Y$, there is a cohomological perfect pairing $\hom^*_{\mathcal{C}}(SX, Y) \otimes \hom(Y,X) \to \mathbb{k}[-n]$.

So, there is an isomorphism $\mathrm{Hom}^*(SX,Y) \cong \mathrm{Hom}^{n-*}(Y,X)^\vee$

ex: V proper smooth algebraic variety over $\mathbb{C}$; then $-\otimes \omega_V$ is a Serre functor for $Coh(V)$: Serre duality implies
$$\omega_V = \Lambda^{top} T^* V$$
$Ext^*(\mathcal{E} \otimes \omega_V, \mathcal{F}) \cong Ext^{n-*}(\mathcal{F}, \mathcal{E})^\vee$.

If V is Calabi-Yau, then (by def) ω_V is trivial $\cong \underline{\mathbb{C}}$, so the Serre functor is trivial (up to shift by n).
"$Coh(V)$ is a CY category of dim n": $\mathrm{Hom}^*(X,Y) \cong \mathrm{Hom}^{n-*}(Y,X)$.

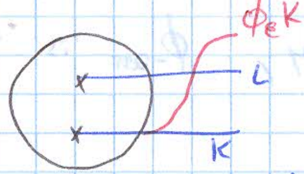
Symplectic setting:

Note that when K, L are compact Lagrangians, or non-compact Lagrangians which are $\pitchfork$ near ∞ (so one can define $CF^*(-,-)$), then note that $HF^*(K,L) \cong HF^{n-*}(L,K)^\vee$.

If $K \pitchfork L$, then $CF^*(K,L)$ and $CF^*(L,K)$ both have the same generators, namely $\{K \pitchfork L\}$.

Why the $(-)^\vee$? If u trajectory between p and q in $CF^*(K,L)$, then u is a trajectory between q and p in $CF^*(L,K)$, for some J.

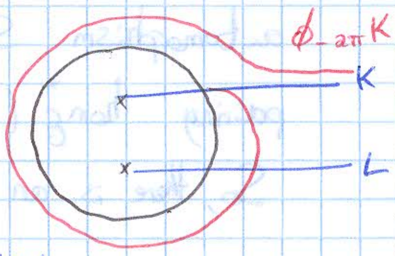
This would suggest that Fukaya categories are always CY categories. Indeed, Fukaya categories of compact Lagrangians are CY categories, in that $\mathrm{Hom}^*_{\mathcal{F}}(K,L) \cong \mathrm{Hom}^{n-*}_{\mathcal{F}}(L,K)^\vee$.
But for LG models, $\mathrm{Hom}_{\mathcal{F}(E,w)}(K,L) := HF^*(\phi_\varepsilon K, L)$, which is asymmetric.

ex: 

~~Note~~ Say the vanishing cycles are both equal to $S \subseteq M$.

Note: $\mathrm{Hom}(K,L) \cong H^*(S)$

But: $\mathrm{Hom}(L,K) = 0$.



Proof of Prop: say we have objects K, L:
Note that $\mathrm{Hom}(\phi_{-2\pi} K, L)$

$= HF^*(\phi_{-2\pi}K, L)$ because $\phi_{-2\pi}K > L$

$\cong HF^*(\phi_{-\varepsilon}K, L)$ because no new intersection points created between $\phi_t K$ and L for $t \in [-2\pi, -\varepsilon]$

$\cong HF^*(K, \phi_\varepsilon L)$, flowing by ϕ_ε

$\cong HF^{n-*}(\phi_\varepsilon L, K)^\vee$ by Poincaré duality

$=: \mathrm{Hom}(L,K)^\vee$.

So, the point is that $-\varepsilon$ suffices, but we pick -2π so it does not depend on K and L. □

So, $\phi_{2\pi}$ is the inverse Serre functor.

In the exact triangle
$$\begin{tikzcd} \mathrm{id} \arrow[r] & \phi_{2\pi} \arrow[dl] \\ \cup_n \arrow[u] & \end{tikzcd}$$
, there is in particular a map $\mathrm{id} \to \phi_{2\pi}$, or equivalently a map $\phi_{-2\pi} \to \mathrm{id}$ "natural transformation from Serre functor to the identity".

This is an extra piece of data determined by the geometry of $\mathcal{F}(E,w)$, beyond what's in $\mathcal{F}(E,w)$:

Theorem [Seidel, Abouzaid-Seidel] The data $(\mathcal{F}(E,w), \{\phi_{-2\pi} \to id\})$ "determine" the wrapped Fukaya category of E (independent of w!).

To first order, the map $id \to \phi_{2\pi}$ is determined by an element $w \in \mathrm{Hom}^0_{\mathcal{F}(E,w)}(\phi_{2\pi}K, K)$ $\forall K \in ob\,\mathcal{F}(E,w)$.

This element is induced by continuation maps:

We can set up a subcomplex which looks like $CF^*(\phi_\varepsilon K, K) \simeq C^*(K)$.

Or directly with continuation maps $\underset{1}{HF^*(\phi_\varepsilon K, K)} \longrightarrow HF^*(\phi_\varepsilon \phi_{2\pi} K, K)$, $1 \longmapsto w$

Theorem (Abouzaid-Seidel) (We are using it as a definition) The subcategory of the wrapped Fukaya category with objects $ob\,\mathcal{F}(E,w)$ is by definition $\mathcal{W} := \mathcal{F}(E,w)[\{id \to \phi_{2\pi}\}^{-1}]$

force all w's to be isomorphisms here

Abouzaid & Seidel also prove that if $K \in ob\,\mathcal{F}(E,w)$, then in $\mathcal{W}$,
$K \simeq \lim(\cdots \to \phi_{4\pi}K \xrightarrow{w} \phi_{2\pi}K \xrightarrow{w} K)$

all isomorphisms

(Need to enlarge $\mathcal{F}(E,w)$ and $\mathcal{W}$ to have arbitrary limits).

Now, in $\mathcal{F}(E,w)$, this diagram $\cdots \to \phi_{4\pi}K \xrightarrow{w} \phi_{2\pi}K \xrightarrow{w} K$ has the property that if one takes hom with L, then multiplying by $w \in \mathrm{Hom}^0(\phi_{2\pi}L, L)$ is a homology isomorphism.

$\mathrm{Hom}(\ldots, L) = \varinjlim_n HF^*(\phi_{2\pi n}K, L)$.

Eventually, this implies that

$$HW^*(K,L) := \mathrm{Hom}_{\mathcal{W}}(K,L) = \varinjlim_n \overbrace{\mathrm{Hom}_{\mathcal{F}(E,w)}(\phi_{2\pi n}K, L)}^{HF^*(\phi_{2\pi n}K,L),\ \text{assuming no} \cap \text{at } \infty}$$

Rem: usually, one takes this as the definition of the Fukaya category, and then the "theorem / definition" above is really a theorem.

We have a triangle

$$\begin{tikzcd} \mathrm{id} \arrow[r] & \phi_{2\pi} \arrow[dl] \\ U \cap K \arrow[u] & \end{tikzcd}$$

, meaning that for every K, we have

$$\begin{tikzcd} K & \phi_{2\pi} K \arrow[l, "w"'] \arrow[dl] \\ U \cap K \arrow[u] & \end{tikzcd}$$

, where w as before.

We can iterate that triangle, to get

$$\begin{tikzcd} K & \phi_{2\pi}^N K \arrow[l] \arrow[dl] \\ \text{some finite complex built of } U(\ldots) \arrow[u] & \end{tikzcd}$$

(Implicitely, we always work in Perf(–) or triangulated hull).

Corollary: if $HW^*(K,K) = 0$, then in $\mathcal{F}(E,w)$, K is split-generated by the image of $U: \mathcal{F}(M) \to \mathcal{F}(E,w)$.

Corollary: if $E = \mathbb{C}^n$ or some other "subcritical" manifold ($\Rightarrow HW^*(K,K) = 0 \;\; \forall K$)

$\Rightarrow$ for such E, and any (E,w), $U: \mathcal{F}(\underset{w^{-1}(p)}{M_p}) \to \mathcal{F}(E,w)$ split-generates.

Proof: $HW^*(K,K) = 0 \iff 1 = 0$ in $HW^*(K,K) = \varinjlim_n HF^*(\phi_{2\pi}^N K, K)$

$\iff 1 = 0$ in $HF^*(\phi_{2\pi}^N K, K)$ for $N \gg 0$

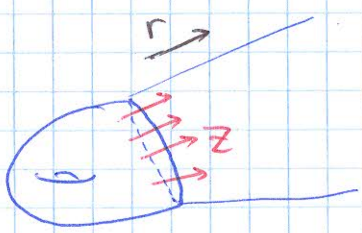
The image of 1 here is exactly $\bar{w}$ (or a product of w's).

So for some $N \gg 0$, we have

$$\begin{tikzcd} K & \phi_{2\pi}^N K \arrow[l, "0"'] \arrow[dl] \\ \text{complex of } U(\ldots) \arrow[u] & \end{tikzcd}$$

So $K \oplus \phi_{2\pi}^N K' \simeq$ complex of $U(\ldots)$

Rem: normally, $HW^*(-,-)$ and $\mathcal{W}(E)$ are defined without reference to $W: E \to \mathbb{C}$ as follows, assuming that E is Liouville:



Z Liouville vector field (near ∞), meaning $d(i_Z \omega) = \omega$.

Z gives a coordinate r on E (near ∞)

Define $HW^*(K,L) = \varinjlim_{\tau > 0} HF^*(K,L;H_\tau)$

OR $= HF^*(K,L;H_{r^2})$

where H_τ is a Hamiltonian which is asymptotically τr near ∞, and H_{r^2} is asymptotically r^2.