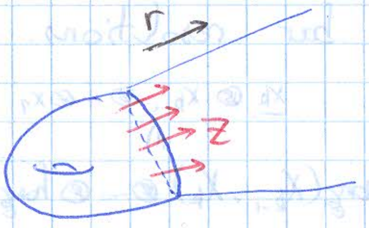
Rem: normally, $HW^*(-,-)$ and $\mathcal{W}(E)$ are defined without reference to $W: E \to \mathbb{C}$ as follows, assuming that E is Liouville:



Z Liouville vector field (near ∞),
meaning $d(i_Z\omega) = \omega$.
Z gives a coordinate r on E (near ∞)

Define $HW^*(K,L) = \varinjlim_{\tau > 0} HF^*(K, L; H_\tau)$

OR $= HF^*(K, L; H_{r^2})$

where H_τ is a Hamiltonian which is asymptotically τr near ∞, and H_{r^2} is asymptotically r^2.

01/06/16

Generation criteria for Fukaya categories [Abouzaid] and for $\mathcal{F}(E,W)$ categories [Abouzaid-Ganatra].

We seek criteria under which a given collection of lagrangians $\{L_i\}_{i=1}^{n}$ (split-) generate the entire Fukaya category.

Decategorified analogy: if one wants to show that a given collection of vectors $\{\vec{v_i}\}_{i=1}^{n}$ spans a vector space V, it suffices to show that in $\mathrm{End}(V) \cong V^* \otimes V$, $id_V = \sum a_{ij}\, \phi_i^* \otimes v_j$

some elements of V^* (ex: v_i^*) if $\exists \langle -,- \rangle$

Then, for any $v \in V$, $v = id_V(v) = \sum_j \left(\sum_i a_{ij}\, \phi_i^*(v)\right) v_j$

§1. Hochschild invariants of A_∞-categories.

To a pair (A,B) where A is an associative algebra and B a bimodule over A, get
* Hochschild homology: $HH_*(A,B) = \mathrm{Tor}_{A\otimes A^{op}}(A,B) = H^*(A \otimes^{L}_{A\otimes A^{op}} B)$
* Hochschild cohomology: $HH^*(A,B) = \mathrm{Ext}_{A\otimes A^{op}}(A,B) = H^*(R\mathrm{hom}_{A\otimes A^{op}}(A,B))$

Shorthand: $HH_*(A) := HH_*(A,A)$ and $HH^*(A) = HH^*(A,A)$. $(*)$

Given an A_∞-category $\mathcal{C}$, we can directly define a chain complex whose cohomology computes HH^*, HH_*, by adapting the explicit complexes in $(*)$ coming from bar resolutions.

Define

denote elements $\underline{x_k} \otimes x_{k-1} \otimes \dots \otimes x_1$

$$CC_*(\mathcal{C},\mathcal{C}) := CC_*(\mathcal{C}) := \bigoplus_{\substack{X_0,\dots,X_k\\ \in \operatorname{ob}\mathcal{C}}} \hom_{\mathcal{C}}(X_k,X_0) \otimes \hom_{\mathcal{C}}(X_{k-1},X_k) \otimes \dots \otimes \hom_{\mathcal{C}}(X_0,X_1)$$

"Hochschild chains"

$$CC^*(\mathcal{C},\mathcal{C}) := \prod_{X_0,\dots,X_k} \hom_{vect}\left(\hom_{\mathcal{C}}(X_{k-1},X_k) \otimes \dots \otimes \hom_{\mathcal{C}}(X_0,X_1), \hom_{\mathcal{C}}(X_0,X_k)\right)$$

"Hochschild cochains"

because $\oplus$ inside $\hom_{vect}$, we took it out

The differential involves summing over ways to apply μ's:

For instance,

$$\delta_{CC_*}(\underline{x_k} \otimes \dots \otimes x_1) := \sum (-1)^* \underline{x_k} \otimes \dots \otimes \mu^i(x_{i+j},\dots,x_{j+1}) \otimes x_j \otimes \dots \otimes x_1$$
$$+ \sum (-1)^* \mu^i(x_s,\dots,x_1,\underline{x_k},\dots,x_{i+j+1}) \otimes x_{i+j} \otimes \dots \otimes x_{s+1} \quad \text{(cyclic)}$$

$\delta_{CC^*}(\psi) := \mu \circ \hat{\psi} \mp \psi \circ \hat{\mu}$, using our previous notation $\hat{}$.

The cohomologies are denoted $HH_*(\mathcal{C})$ and $HH^*(\mathcal{C})$; graded if $\mathcal{C}$ is.

More generally, can take $HH_*(\mathcal{C},\mathcal{B})$, where $\mathcal{B}$ is an A_∞-bimodule over $\mathcal{C}$, i.e. a bilinear functor $\mathcal{B}: \mathcal{C}^{op} \times \mathcal{C} \to Ch_{\Bbbk}$

$$CC_*(\mathcal{C},\mathcal{B}) = \bigoplus \mathcal{B}(X_k,X_0) \otimes \hom_{\mathcal{C}}(X_{k-1},X_k) \otimes \dots \otimes \hom_{\mathcal{C}}(X_0,X_1).$$

§2. Open-closed and closed-open maps.

Fix a field $\Bbbk$ ex: $\Bbbk = \Lambda_{\mathbb{C}}$, q formal variable, $q \in \Bbbk$

Say X is a compact symplectic manifold (for instance monotone, but take any other setting where all structures are defined).

↳ $\mathcal{F}(X)$ Fukaya category ($\mathbb{Z}$-graded if $2c_1(X) = 0$; otherwise $\mathbb{Z}/2$ or $\mathbb{Z}/2k$-graded).

↳ $QH^*(X)$ quantum cohomology; same grading as above. As a vector space, $QH^*(X) := H^*(X;\Bbbk)$ (with grading collapsed)

$(-,-)_X : QH^*(X)^{\otimes 2} \longrightarrow \Bbbk$, $(\alpha,\beta)_X := \int_X \alpha \cup \beta$

and w.r.t. $(-,-)_X$, $\star : (QH^*(X))^{\otimes 2} \to QH^*(X)$ equivalent to the data of "3-points functions"

$(\alpha,\beta,\gamma)_X := (\alpha \star \beta, \gamma)_X$, counting

weighted by $q^{\omega(u)}$.

[Seidel, Fooo, Abouzaid]: there are geometric maps

$\mathcal{OC} : HH_{*-n}(\mathcal{F}(X)) \longrightarrow QH^*(X)$

$\mathcal{CO} : QH^*(X) \longrightarrow HH^*(\mathcal{F}(X))$.

Proposition: * $\mathcal{CO}$ is a ring map

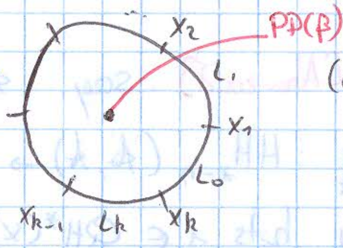
* $\mathcal{OC}$ is a $QH^*(X)$-module map, where the QH^* module structure on HH_* is induced by $\mathcal{CO}$ and the HH^*-module structure on HH_* (non-commutative).

It is broadly expected that $(\mathcal{OC}, \mathcal{CO})$ the "Hochschild calculus" with standard operations on $QH^*(X)$.

How to define these maps, broadly?

* Given $\underline{x_k} \otimes x_{k-1} \otimes \cdots \otimes x_1$, to define $\mathcal{OC}$, it suffices to specify:

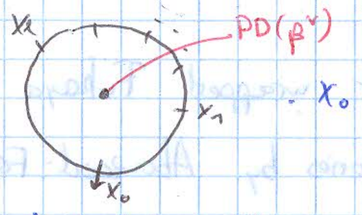
$(\mathcal{OC}(\underline{x_k} \otimes x_{k-1} \otimes \cdots \otimes x_1), \beta)_X := \#$



(weighted count by $q^{\omega(u)}$).

* Given $\beta \in QH^*(X)$,

$\mathcal{CO}(\beta)(x_\ell, \ldots, x_1) = \sum_{x_0} \#$



We needed to choose cycles, but the result is independent of choices.

Proposition: these chain level $\mathcal{OC}$ and $\mathcal{CO}$ descend to cohomology; call the cohomology level maps $\mathcal{OC}$ and $\mathcal{CO}$ too.

Proof: analyze codim 1 breaking. $\square$

Rem. when X is monotone, both $QH^*(X)$ and $\mathcal{F}(X)$ decompose into summands indexed by $w \in \mathbb{k}$, where w is an eigenvalue of $(c_1(X) \star -) : QH^*(X) \to QH^*(X)$.
Call $QH^*(X)_w$, $\mathcal{F}_w(X)$ the corresponding summands.

Proposition: [Ritter–Smith, see also Sheridan] The maps $\mathcal{OC}$ and $\mathcal{CO}$ "respect these decompositions".

Proposition: $\mathcal{OC}$ and $\mathcal{CO}$ are "linear dual" in the following sense: have $(-,-)_X : QH^*(X) \xrightarrow{\simeq} QH^*(X)^\vee$, and $\mathcal{F}(X)$ (when X is compact, or rather when L's are) is a "weak C-Y category" (some refinement of $HF^*(K,L) \cong HF^*(L,K)^\vee[-n]$, which implies $HH_*(\mathcal{F}(X))^\vee[-n] \cong HH^*(\mathcal{F}(X))$). We have

$$\begin{array}{ccc} QH^*(X) & \xrightarrow{\mathcal{CO}} & HH^*(\mathcal{F}(X)) \\ {\scriptstyle \alpha \mapsto (\alpha,-)_X} \downarrow & \circlearrowleft & \downarrow CY_* \\ QH^*(X)^\vee & \xrightarrow[\mathcal{OC}^\vee]{} & HH_*(\mathcal{F}(X))^\vee \end{array}$$

§3. Abouzaid's generation criterion.

Theorem [Abouzaid] say $\mathcal{A} \subseteq \mathcal{F}(X)$ full subcategory, have
$$\mathcal{OC}|_{\mathcal{A}} : HH_{*-n}(\mathcal{A},\mathcal{A}) \to HH_{*-n}(\mathcal{F},\mathcal{F}) \longrightarrow QH^*(X)$$
If $\mathcal{OC}|_{\mathcal{A}}$ hits $1 \in QH^*(X)$, then $\mathcal{A}$ split-generates $\mathcal{F}(X)$.

(originally for wrapped Fukaya categories, implemented for compact Fukaya categories by Abouzaid–FOOO, Ritter–Smith, Sheridan, Perutz–Sheridan).

If $\mathcal{A} \subseteq \mathcal{F}_w(X)$ and $\mathcal{OC}|_{\mathcal{A}}$ hits $pr_{QH^*(X)_w}(1)$, then $\mathcal{A}$ split-generates $\mathcal{F}_w(X)$.
"Works one summand at a time".

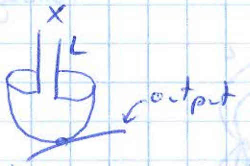
Sketch of proof: "annulus argument" (or Cardy condition)

Baby case: note that there is a map $HF^*(L,L) \to HH_*(\mathcal{A})$ for any $L \in \mathrm{ob}\,\mathcal{A}$ ($\hom_{\mathcal{A}}(L,L)$ subcomplex of $CC_*(\mathcal{A})$).

There is also a map $HH^*(\mathcal{A},\mathcal{A}) \to HF^*(K,K)$ for any $K \in \mathrm{ob}\,\mathcal{A}$ ($\hom_{\mathcal{A}}(K,K)$ quotient complex of $CC^*(\mathcal{A},\mathcal{A})$).

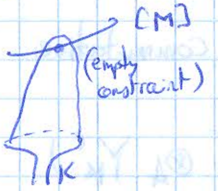
Suppose $\mathcal{OC}|_{\mathcal{A}} : HF^*(L,L) \to HH_*(\mathcal{A},\mathcal{A})$ hits 1
$L \in \mathcal{A}$

$$\begin{tikzcd} HF^*(L,L) \arrow[r] \arrow[dr] & HH_*(\mathcal{A},\mathcal{A}) \arrow[d] \\ & QH^*(X) \end{tikzcd}$$

counts 

For an arbitrary $K \in \mathcal{F}(X)$, have $\mathcal{CO}^0 : QH^*(X) \to HF^*(K,K)$.

Claim: this maps $1 \mapsto 1$ always

(exercise  $\simeq$  "unstable count")

Claim: for any L, K, $\exists$ a comm. diagram

$$\begin{tikzcd} HF^*(L,L) \arrow[r, "\Delta_*"] \arrow[d, "\mathcal{OC}^0"] & HF^*(L,K) \otimes HF^*(L,K) \arrow[d, "{[\mu^2]}"] \\ QH^*(X) \arrow[r, "\mathcal{CO}^0"] & HF^*(K,K) \end{tikzcd}$$

$$1 \longmapsto 1_K$$

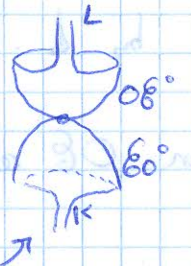
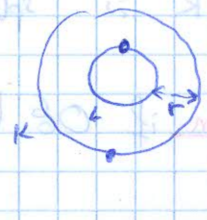
(Δ_* is "coprod", a "new operation".

So, if $\mathcal{OC}^0_L$ hits 1, then for any K,

$HF^*(L,K) \otimes HF^*(K,L) \xrightarrow{\mu^2} HF^*(K,K)$ hits 1_K.

meaning that in $H^*\mathcal{F}(X)$, $\begin{tikzcd} K \arrow[r, "a"] \arrow[rr, bend right, "\mathrm{id}_K"'] & L \arrow[r, "b"] & K \end{tikzcd}$, so

any K is a summand of L.

Proof of 2nd claim:  $\rightsquigarrow$ give , because

$\mathcal{CO}^0 \circ \mathcal{OC}^0$ counts ↗

$$\mathcal{OC}(x) = \sum_{\beta_i \in H^*(X) \text{ basis}} \mathcal{OC}_{\beta_i^\vee}(x) \cdot \beta_i$$

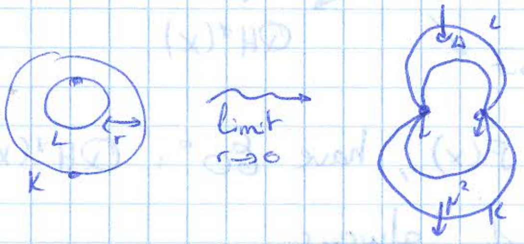
$$\mathcal{CO}(\beta_i) = \sum_{x \in CF^*(K,K)} (\dots)$$

$$\mathcal{CO} \circ \mathcal{OC}(x) = \sum_{\beta_i \in H^*(X)} \mathcal{OC}_{\beta_i^\vee}(x)\, \mathcal{CO}(\beta_i) \qquad \text{(use fact } (\Delta) = \textstyle\sum_i \beta_i^{*} \otimes \beta_i\text{)}$$

$$= \sum (\mathcal{OC}, \mathcal{CO})_{\beta_i^* \otimes \beta_i}$$

$$= (\mathcal{OC}, \mathcal{CO})_\Delta \qquad \text{(homologically)}$$

And



More general case: there's a commutative diagram [Abouzaid]:

$$\begin{tikzcd}
HH_*(\mathcal{A},\mathcal{A}) \arrow[r] \arrow[d, "\mathcal{OC}"'] & H^*(\mathcal{Y}_K^r \otimes_{\mathcal{A}} \mathcal{Y}_K^l) \arrow[d, "{[\mu]}"] \\
QH^*(X) \arrow[r, "\mathcal{CO}^0"'] & HF^*(K,K)
\end{tikzcd}$$

any object in $\mathcal{F}(X)$

where $\mathcal{Y}_K^r \otimes_{\mathcal{A}} \mathcal{Y}_K^l$ is a chain complex of the form

$$\bigoplus_{X_0,\dots,X_k} \hom_{\mathcal{C}}(X_k, K) \otimes \hom_{\mathcal{A}}(X_{k-1}, X_k) \otimes \dots \otimes \hom_{\mathcal{A}}(X_0, X_1) \otimes \hom_{\mathcal{C}}(K, X_0)$$

This diagram implies that if $\mathcal{OC}$ hits 1, then $[\mu]$ hits 1 for any K: (let $i : \mathcal{A} \hookrightarrow \mathcal{F}(X)$)

$\Rightarrow \hom(K, i_* i^* X) \xrightarrow{\mu} \hom(K,K)$ hits 1...

$\Rightarrow 1 = 0$ in $H^* \hom_{\mathcal{F}/\mathcal{A}}(K,K)$

$\Leftrightarrow K \simeq 0$ in $\mathcal{F}/\mathcal{A}$

$\Leftrightarrow K$ is split-generated by $\mathcal{A}$. $\square$

Theorem: if $\mathcal{OC}$ hits 1, then $\mathcal{OC}$ and $\mathcal{CO}$ are isomorphisms.

There are many instances in which one can verify this generation criterion. By PD of $\mathcal{OC}$ and $\mathcal{CO}$, it suffices to show $\mathcal{CO}: QH^*(X) \to HH^*(\mathcal{A})$ is injective: it implies $\mathcal{OC}$ is surjective, hence hits 1.

There are many cases in which a given $QH^*_w(X)$ is rank 1: $= \Bbbk \langle \underbrace{pr_{QH^*_w}(1)}_{e_w} \rangle$

In this case, if $\mathcal{A} \subseteq \mathcal{F}_w(X)$, and $\mathcal{A}$ has any L with $HF^*(L,L) \neq 0$, $\mathcal{A}$ satisfies the generation criterion ("semi-simple case")

Indeed, $\mathcal{CO}_w$ is injective:

$$QH^*_w \longrightarrow HH^*(\mathcal{A},\mathcal{A}) \longrightarrow HF^*(L,L)$$

$$1_w \longmapsto 1 \longmapsto 1 \neq 0.$$

ex: $\mathbb{P}^1$, or more generally $\mathbb{P}^n$ ($QH^*(X)$ splits into rank 1 summands)

The Clifford torus with all its local systems generate, for instance.

There are other cases in which one can deduce "automatic generation":

example: **Theorem [Ganatra]** Say

(1) $\mathcal{A}$ is "homologically smooth" (some condition only depending on $\mathcal{A}$ as an A_∞-category, satisfied if $perf(\mathcal{A}) \simeq Coh(Y)$ or $MF(Y,w)$).

(2) $rk\, HH^0(\mathcal{A}) \geq rk\, QH^0(X)_{(w)}$ ← if $\mathcal{A} \subseteq \mathcal{F}_w(X)$

Then, $\mathcal{A}$ split-generates.

Can apply this to other cases, such as Fano varieties, Fano hypersurfaces in $\mathbb{P}^n$, using computations of [Cho, Cho-Oh, A-FOOO] and [Smith, Sheridan].

Returning to LG models (E, w), there is a map

$$HH_*(\mathcal{F}(E,w), \mathcal{B}_{\phi_{2\pi}}) \xrightarrow[\mathcal{OC}_w]{} HF^*(E,w)$$

Theorem [Abouzaid-Ganatra] if $\mathcal{OC}_{w|\mathcal{A}}$ hits 1, $\mathcal{A}$ split-generates.

Expectation: $\mathcal{OC}_w$ is always an isomorphism, at least when W is a Lefschetz fibration (true with 1 critical point,...)