Returning to LG models (E, w), there is a map
$$HH_*(\mathcal{F}(E,w), \mathcal{B}_{\phi_{2\pi}}) \xrightarrow[\mathcal{OE}_W]{} HF^*(E,w)$$

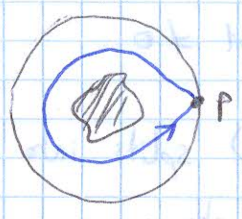
Theorem [Abouzaid-Ganatra] if $\mathcal{OE}_{W|\mathcal{A}}$ hits 1, $\mathcal{A}$ split-generates.

Expectation: $\mathcal{OE}_W$ is always an isomorphism, at least when W is a Lefschetz fibration (true with 1 critical point, ...)

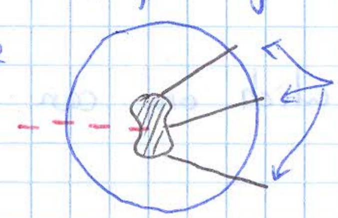
03/06/16

Generating Fukaya categories of LG models

Setup: (E^{2n+2}, W) symplectic LG model: $W: E \to \mathbb{C}$, with $M := W^{-1}(p)$ general fiber and $\mu: M \to M$ symplectic monodromy induced by



↳ $\mathcal{F}(E,W)$ has for objects Lagrangian branes, such that $W(L)$ looks like



Last time: generation criterion: X^{2n+2} compact symplectic, we can construct $\mathcal{OC}: HH_{*-n}(\mathcal{F}(X)) \to QH^*(X)$

Theorem: [Abouzaid-Ganatra] If $\mathcal{OC}|_{\mathcal{A}}$ hits $1 \in QH^*(X)$, then $\mathcal{A}$ split-generates $\mathcal{F}(X)$.

Theorem: under the above hypotheses, $\mathcal{OC}$ and
$$\mathcal{CO}: QH^*(X) \to HH^*(\mathcal{F}(X))$$
are isomorphisms, preserving various structures.

Want: a similar criterion to this one, for $\mathcal{F}(E,w)$

Application: (some still in progress, or conjectural)

* Show a basis of thimbles (or some other collection of Lagrangians) split-generates.
* Give a geometric description of the Hochschild invariants of $\mathcal{F}(E,w)$ $\rightsquigarrow$ new invariants of (E,w), which can be used to detect various phenomena (symplectic invariants of singularities of polynomials).

Problems. (1) $HH_*(\mathcal{F}(E,w))$ is the wrong place for a generation criterion.
(homology)

ex: for $\mathcal{A} \subseteq \mathcal{F}(E,w)$ a basis $\{\Delta_1, \dots, \Delta_k\}$ of thimbles, we showed that a minimal model of $\mathcal{A}$ looks like:

$$\hom_{\mathcal{A}}(\Delta_i, \Delta_j) = \begin{cases} HF^*(V_i, V_j) & i<j \\ \mathbb{k} & i=j \\ 0 & i>j \end{cases}$$

⚠ need cyclic chains in CC

(*) $\Rightarrow HH_*(\mathcal{A}) = \bigoplus_{i=1}^{n} \mathbb{k}$ ↪ corresponds to $\hom_{\mathcal{A}}(\Delta_i, \Delta_i)$

We get this because for $\mathcal{A}$ strictly unital augmented, there is a reduced version of CC

$$CC^{red}(\mathcal{A}) = \bigoplus_{X_0,\dots,X_k} \hom_{\mathcal{A}}(X_k, X_0) \otimes \overline{\hom_{\mathcal{A}}(X_{k-1}, X_k) \otimes \dots \otimes \hom_{\mathcal{A}}(X_0, X_1)}$$

in reduced thing

The only place we can have a unit is the 1st one, hence (*).

(2) "Closed string invariant" of (E,w) (analogue of $QH^*(X)$). Ideally, it should be a unital ring, meaning it has a "1", and maybe it should be $\cong HH^*(\mathcal{F}(E,w))$ the cohomology (we just saw it can't be the homology), which is a unital ring too.

Conjecture of Seidel (2001): let w Lefschetz, $\mathcal{A}$ any basis of thimbles; there should be a LES

$$\begin{tikzcd} HH^*(A,A) \arrow[dr, "{[1]}"'] & & H^*(E) \arrow[ll] \\ & \underline{HF^*(\mu)} \arrow[ur] & \end{tikzcd}$$

$\mu : M \to M$

↳ "fixed point Floer cohomology, generated by the fixed points of $\mu : M \to M$".

ex: $E = \mathbb{C}^{n+1}$ $\quad W = \sum z_i^2$

$H^*(E) = H^*(pt) = \mathbb{k}$

$HF^*(\mu) = 0$ "a Dehn twist has no fixed points"

$HH^*(\mathrm{End}(\Delta)) = HH^*(\mathbb{k}) \cong \mathbb{k}$

$\to$ Can't put $H^*(M)$ instead of $H^*(E)$: wouldn't work.

Rem: conjecture based on a lot of computations; now we have more geometric reasons.

<u>Solutions</u>:

(2) [Abouzaid-Ganatra] we can define a unital <u>ring</u> $HF^*(E,W) \ni 1$, with maps $H^*(E) \longrightarrow HF^*(E,W) \longrightarrow SH^*(E)$.

([Seidel, earlier]: proposed a model for $HF^*(E,W)$ as a <u>group</u>)

Furthermore, the map $H^*(E) \to HF^*(E,W)$ fits into a LES with $HF^*(\mu)$.

(1) In the compact case, a consequence of satisfying the generation criterion is an isomorphism $HH_*(\mathcal{F}(X)) \cong HH^*(\mathcal{F}(X))$. This is not true for any category; it is a manifestation of a duality passessed by $\mathcal{F}(X)$, analogous to PD, called a (type of) "Calabi-Yau structure".

According to HMS, often a closed Calabi-Yau manifold X is mirror to a closed Calabi-Yau manifold Y:

$\mathrm{Fuk}(X) \simeq \mathrm{Coh}(Y)$, after taking $\mathrm{Perf}(-)$.

Note that Y being CY $\iff \exists$ a non-vanishing holomorphic volume form $vol_Y \in H^0(Y, \Omega_Y^{top})$. If such a vol_Y exists, contraction induces an isomorphism

$$\begin{array}{ccc} i_{(\cdot)} vol_Y : H^*(Y, \Lambda^* TY) & \xrightarrow{\cong} & H^*(Y, \Omega_Y^{\bullet}) \\ \wr\!\wr \text{ (essentially HKR + [Kontsevich, Toën, ...])} & & \wr\!\wr \text{ (also)} \\ HH^*(Coh(Y)) & \cong & HH_*(Coh(Y)) \end{array}$$

get this

(E, W) arise frequently as mirrors to Fano manifolds (non CY); HMS expects that for some pair $(Y, (E,W))$,
Fano, symplectic LG

$$Coh(Y) \cong \mathcal{F}(E, W), \text{ after } Perf(-).$$

Note that $\nexists$ $vol_Y \in H^0(Y, \Omega_Y^n)$, but $\exists$ $vol_Y \in H^0(Y, \Omega_Y^n \otimes K_Y^{-1})$, inducing an isomorphism

$$\begin{array}{ccc} i_{(\cdot)} vol_Y : H^0(Y, \Lambda^\bullet TY) & \xrightarrow{\cong} & H^0(Y, \Omega_Y^\bullet \otimes K_Y^{-1}) \\ \wr\!\wr & & \wr\!\wr \\ HH^*(Coh(Y)) & & HH_*(Coh(Y), (- \otimes \omega_Y^{-1})) \end{array}$$

the "Serre inverse functor" $- \otimes \omega_Y^{-1} : Coh(Y) \circlearrowleft$

Rem. If $\mathcal{C}$ A_∞-category, $\mathcal{F} : \mathcal{C} \circlearrowleft$ a functor, there exists a bimodule $\mathcal{B}_\mathcal{F}$ "the graph of $\mathcal{F}$" with the prescription, for $X, Y \in ob\,\mathcal{C}$,

$$\mathcal{B}_\mathcal{F}(X, Y) := hom_\mathcal{C}(\mathcal{F}X, Y),$$

and we define $HH_*(\mathcal{C}, \mathcal{F}) := HH_*(\mathcal{C}, \mathcal{B}_\mathcal{F})$ (roughly defined last time)

$\hookrightarrow$ Solution to (1): we should look at

$$HH_*(\mathcal{F}(E,W), \mathcal{B}_{\phi_{2\pi}}), \text{ where } \phi_{2\pi} \text{ is the "once wrapping"}$$

~~[Abouzaid, Ganatra]~~:

Results (Abouzaid, Ganatra):

(1) Can build an open-closed map

$$\mathcal{OC}_W : HH_{*-n}(\mathcal{F}(E,W), \mathcal{B}_{\phi_{2\pi}}) \longrightarrow HF^*(E,W).$$

To first order, get a map $HF^*(\phi_{2\pi+\varepsilon} L, L) \to HF^*(E,W)$ $\forall L$.

(2) Can build a map

$$\mathcal{CO}_W : HF^*(E,W) \longrightarrow HH^*(\mathcal{F}(E,W)).$$

To first order, get a map $HF^*(E,W) \longrightarrow HF^*(\phi_\varepsilon K, K)$ $\forall K$
$1 \longmapsto 1$

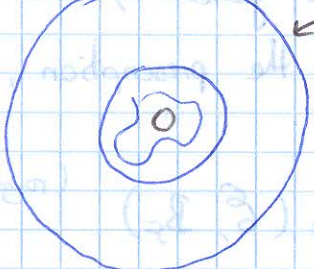
Rem: the asymmetry: need to twist for $\mathcal{OC}$, but not $\mathcal{CO}$.

Theorem: let $\mathcal{A} \subseteq \mathcal{F}(E,W)$ be a subcategory.
If $(\mathcal{OC}_W)|_{\mathcal{A}}$ hits $1 \in HF^*(E,W)$, then $\mathcal{A}$ split-generates.

Expect also: then, $\mathcal{OC}_W$ and $\mathcal{CO}_W$ are isomorphisms.

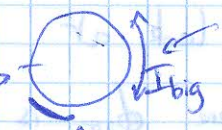
A word on defining $HF^*(E,W)$. It arises as a Hamiltonian Floer homology $HF^*(H) \cong HF^*(\Gamma_{\phi_H}, \Delta \subseteq E \times E)$, generated by the fixed points of ϕ_H^1, i.e. the time-1 orbits.

Idea: pull back from $\mathbb{C}$ (via W) a Hamiltonian of the following form:



near ∞, should be a function of the form
$X_t(\theta) . h(r)$
↳ rotation type flow
↳ time-dependent, $t \in S^1$.

satisfying



on this angular sector, flow rotates by $\geq 2\pi$ and $< 4\pi$

rotate ε here

everywhere else, rotates strictly less than 2π

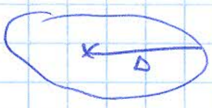
on I_{small}, rotate $< \pi$ ($\to$ does not cross θ).

Call $\mathcal{X}_W$ this class of Hamiltonians.

<u>Claim:</u> we can define HF^*(this Hamiltonian), and it has an algebra structure!
(because we can compose $\phi^1_{H_1} \circ \phi^1_{H_2}$ and it will still be in this class, if roughly "$I^{H_1}_{big} \cap I^{H_2}_{big} = \phi$").

* Get $\mathcal{OE}_W$ map by noting that if $D(L) \subseteq I_{big} \subseteq (-\pi, \pi)$, then $\phi_H(L) \simeq \phi_{2\pi}(L)$ if $H \in \mathcal{X}_W$.
* Get $\mathcal{EO}$ map: if $D(K) \in I_{small} \subseteq (-\pi,\pi)$, then $\phi^1_H(K) \simeq \phi_\pi(K) \simeq K$ in $\mathcal{F}(E,w)$.

It is easy to prove ~~a criterion: that~~ if W has one Lefschetz critical point that Δ satisfies (the criterion; to see that,) transplant in this local model
$\mathbb{C}^{n+1} \xrightarrow{\Sigma z_i^2} \mathbb{C}$



$$\begin{array}{ccc}
HF^*(\phi_{2\pi}\Delta, \Delta) & \xrightarrow{\Delta_*} & HF^*(\phi_{2\varepsilon}\Delta, \phi_\varepsilon \Delta) \otimes HF^*(\phi_\varepsilon \Delta, \Delta) \\
\downarrow \mathcal{OE}_W & & \downarrow \mu^2 \\
HF^*(E,w) & \xrightarrow{\mathcal{EO}_W} & HF^*(\phi_{2\varepsilon}\Delta, \Delta) \\
\wr\| & 1 \longmapsto 1 & \\
H^*(E) = \mathbb{k} & &
\end{array}$$

<u>Claim:</u> μ^2 and Δ_* are both isos, and hit 1.