Cyclic homology, S^1-equivariant Floer cohomology, and Calabi-Yau structures

Sheel Ganatra

Abstract. We construct geometric maps from the cyclic homology groups of the (compact or wrapped) Fukaya category to the corresponding S^1-equivariant (Floer/quantum or symplectic) cohomology groups, which are natural with respect to all Gysin and periodicity exact sequences and are isomorphisms whenever the (non-equivariant) open-closed map is. These *cyclic open-closed maps* give (a) constructions of geometric smooth and/or proper Calabi-Yau structures on Fukaya categories (the latter is equivalent, in characteristic 0, to the existence of a cyclic A_∞ model) and (b) a purely symplectic proof of the non-commutative Hodge-de Rham degeneration conjecture for smooth and proper subcategories of Fukaya categories. Further applications of cyclic open-closed maps, to counting curves in mirror symmetry and to comparing topological field theories, are the subject of joint projects with Perutz-Sheridan [**GPS2**, **GPS1**] and Cohen [**CG**].

1. Introduction

This paper concerns the compatibility between chain level S^1 actions arising in two different types of Floer theory on a symplectic manifold. The first of these $C_{-*}(S^1)$[1] actions is induced geometrically on the *Hamiltonian Floer homology chain complex* $CF^*(M)$, formally a type of Morse complex for an action functional on the free loop space, through rotating free loops. The homological action of $[S^1]$ is known as the *BV operator* $[\Delta]$, and the $C_{-*}(S^1)$ action can be used to define S^1*-equivariant Floer homology theories* — see e.g., [**S4**, **BO**][2]. The second $C_{-*}(S^1)$ action lies on the *Fukaya category*, and has discrete or combinatorial origins, coming from the hierarchy of compatible cyclic $\mathbb{Z}/k\mathbb{Z}$ actions on cyclically composable chains of morphisms between Lagrangians. A (categorical analogue of a) fundamental observation of Connes', Tsygan, and Loday-Quillen that such a structure, which exists on any category $\mathcal{C}$, can be packaged into a $C_{-*}(S^1)$ action on the *Hochschild homology chain complex* $\mathrm{CH}_*(\mathcal{C})$ of the category (see e.g., [**C1**, **T2**, **LQ**, **M1**, **K2**]) The associated operation of multiplcation by (a cycle representing) $[S^1]$ is frequently called the *Connes' B operator* B, and the corresponding S^1-equivariant homology theories are called *cyclic homology groups*.

A relationship between the Hochschild homology of the Fukaya category and Floer homology is provided by the so-called *open-closed string map* [**A**]

$$\mathcal{OC} : \mathrm{CH}_*(\mathcal{F}) \to CF^{*+n}(M). \tag{1.1}$$

Our main result is about the compatibility of $\mathcal{OC}$ with $C_{-*}(S^1)$ actions. Namely, we prove that $\mathcal{OC}$ can be made (coherently homotopically) $C_{-*}(S^1)$-equivariant:

The author was partially supported by the National Science Foundation through a postdoctoral fellowship — grant number DMS-1204393 — and agreement number DMS-1128155. Any opinions, findings and conclusions or recommendations expressed in this material are those of the author(s) and do not necessarily reflect the views of the National Science Foundation.

[1] We use a cohomological grading convention in this paper, so singular chain complexes are *negatively graded*.

[2] Sometimes S^1-equivariant Floer theory is instead defined as Morse theory of an action functional on the S^1-Borel construction of the loop space. For a comparison between these two definitions, see [**BO**].

Theorem 1. *The map $\mathcal{OC}$ admits a geometrically defined 'S^1-equivariant enhancement', to an A_∞ homomorphism of $C_{-*}(S^1)$-modules, $\widetilde{\mathcal{OC}} \in \mathrm{RHom}^n_{C_{-*}(S^1)}(\mathrm{CH}_*(\mathcal{F}), \mathrm{CF}^*(M))$.*

Remark 1. Theorem 1 implies (but is not implied by) the homological fact that $[\mathcal{OC}]$ intertwines homological actions of $[S^1]$. This is proven as a stepping stone to Theorem 1 in Theorem 5 below.

Remark 2. Note that $\mathcal{OC}$ typically does not have the property of strictly intertwining the $C_{-*}(S^1)$ actions. In particular, the homomorphism $\widetilde{\mathcal{OC}}$ involves *extra data* recording coherently higher homotopies between the two $C_{-*}(S^1)$ actions. This explains our use of the term 'enhancement'.

Remark 3. It can be shown using usual invariance arguments that this enhancement is canonical: any two such geometrically defined enhancements are homotopic.

To explain the consequences of Theorem 1 to cyclic homology and equivariant Floer homology, recall that there are a variety of S^1-*equivariant homology* chain complexes (and homology groups) that one can associate functorially to an A_∞ $C_{-*}(S^1)$ module P. For instance, denote by

$$P_{hS^1},\ P^{hS^1},\ P^{Tate} \tag{1.2}$$

the *homotopy orbit complex*, *homotopy fixed point complex*, and *Tate complex* constructions of P, described in §2.2. When applied to the Hochschild complex $\mathrm{CH}_*(\mathcal{C}, \mathcal{C})$, the constructions (1.2) by definition recover complexes computing *(positive) cyclic homology*, *negative cyclic homology*, and *periodic cyclic homology* groups of $\mathcal{C}$ respectively (see §3.1). Similarly $H^*(CF^*(M)_{hS^1})$ is the S^1-*equivariant Floer cohomology* studied (for the symplectic homology Floer chain complex) in [**BO**]. The groups $H^*(CF^*(M)^{hS^1})$ and $H^*(CF^*(M)^{Tate})$ have also been studied in recent work in Floer theory [**S8**] . Functoriality of the constructions (1.2) and homotopy invariance properties of $C_{-*}(S^1)$ modules (see Cor. 3 and Prop. 2) immediately implies the result announced in the abstract:

Corollary 1. *Let $HF^{*,+/-/\infty}_{S^1}(M)$ denote the (cohomology of the) homotopy orbit complex, fixed point complex, and Tate complex construction applied to $CF^*(M)$, and let $\mathrm{HC}^{+/-/\infty}(\mathcal{C})$ denote the corresponding positive/negative/periodic cyclic homology groups. Then, $\widetilde{\mathcal{OC}}$ induces* cyclic open-closed maps

$$[\widetilde{\mathcal{OC}}^{+/-/\infty}] : \mathrm{HC}^{+/-/\infty}_*(\mathcal{F}) \to HF^{*+n,+/-/\infty}_{S^1}(M), \tag{1.3}$$

naturally compatible with respect to the various periodicity/Gysin exact sequences, which are isomorphisms whenever $\mathcal{OC}$ is. □

The map (1.1) is frequently an isomorphism, allowing one to recover in these cases closed string Floer/quantum homology groups from open string, categorical ones [**BEE**, **G1**, **GPS2**, **AFO**$^+$]. In such cases, Theorem 1 and Corollary 1 allows one to further categorically recover the $C_{-*}(S^1)$ action on $CF^*(M)$ and associated equivariant homology groups.

Remark 4. There are other S^1-equivariant homology functors, and our results apply tautologically to them as well. For instance, consider the contravariant functor $P \mapsto (P_{hS^1})^\vee$; when applied to $\mathrm{CH}_*(\mathcal{C})$ this produces the *cyclic cohomology* chain complex of $\mathcal{C}$.

We have been deliberately vague about which Fukaya category and which Hamiltonian Floer homology groups Theorem 1 applies to, as it applies in several geometric settings. The simplest of these are

(1) If M is compact, $\mathcal{F}$ is the usual Fukaya category of compact Lagrangians (or a summand thereof). $CF^*(M)$, the Hamiltonian Floer complex of any Hamiltonian, is quasi-isomorphic to the *quantum cohomology* ring with its trivial $C_{-*}(S^1)$ action.
(2) If M is non-compact and Liouville, one could take $\mathcal{F} = \mathcal{W}$ to be the *wrapped Fukaya category* and $CF^*(M) = SC^*(M)$ to be the *symplectic cohomology ring* with its (usually non-trivial) $C_{-*}(S^1)$ action,

(3) If M is non-compact and Liouville, one could take $\mathcal{F} \subset \mathcal{W}$ to be the *Fukaya category of compact Lagrangians*. In this case, the map $\mathcal{OC}$ factors through $CF^*(M) = H^*(M, \partial^\infty M)$ the *relative cohomology group* with its trivial $C_{-*}(S^1)$ action. In fact, $\mathcal{OC}$ further factors through the *symplectic homology chain complex* $SC_*(M) = (SC^*(M))^\vee$.

For example, in case (2) above, when the relevant $[\mathcal{OC}]$ map is an isomorphism, Corollary 1 computes various S^1-equivariant symplectic cohomology groups (such as the usual equivariant symplectic cohomology $SH^*_{S^1}(M)$, which is the homology of the homotopy orbits complex $SC^*(M)_{hS^1}$) in terms of cyclic homology groups of the wrapped Fukaya category.

To keep this paper a manageable length, we implement the map $\widetilde{\mathcal{OC}}$ and prove Theorem 1 in the technically simplest settings, when M is either compact and monotone or its Lagrangians do not bound pseudoholomorphic discs, or when M is Liouville (non-compact, exact, convex at ∞). However, our methods are entirely are orthogonal to the usual analytic difficulties faced in constructing Fukaya categories and open-closed maps in general contexts, and we expect they should extend immediately to other settings. For instance, in the setting of relative Fukaya categories of compact projective Calabi-Yau manifolds, an adapted version of our construction will appear in joint work with Perutz-Sheridan [**GPS1**].

Remark 5. There are other settings in which Fukaya categories are well-studied, for instance Fukaya categories of Lefschetz fibrations (and more general LG models), or more generally partially wrapped Fukaya categories. We do not discuss these situations in our paper, but expect suitable versions of Theorem 1 to hold in such settings too.

Remark 6. One can consider variations on Theorem 1. As a notable example, let M denote a (noncompact) Liouville manifold, and $\mathcal{F}$ the Fukaya category of compact Lagrangians in M. Then there is a non-trivial refinement of the map $\mathrm{HH}_*(\mathcal{F}) \to H^*(M, \partial M)$, which can be viewed as a pairing $\mathrm{HH}_*(\mathcal{F}) \times H^*(M) \to \mathbf{k}$, to a pairing

$$\mathrm{CH}_*(\mathcal{F}) \otimes SC^*(M) \to \mathbf{k}.$$

(note symplectic cohomology does not satisfy Poincaré duality in any sense, so this is *not* equivalent to a map to symplectic cohomology). Our methods also imply that this pairing admits an S^1-equivariant enhancement, with respect to the diagonal $C_{-*}(S^1)$ action on the left and the trivial action on the right. Passing to adjoints, we obtain cyclic open closed maps from S^1-equivariant symplectic cohomology to cyclic *cohomology* groups of $\mathcal{F}$, and from cyclic homology of $\mathcal{F}$ to equivariant symplectic *homology*. See §5.6.2 for a more details.

Beyond categorically computing equiariant Floer cohomology groups, we describe below two immediate applications of Theorem 1 to the structure of Fukaya categories.

Remark 7. We anticipate additional concrete applications of Theorem 1 and its homological shadow, Theorem 5. For instance, one can study the compatibility of open-closed maps with *dilations* in the sense of [**SS**], which are elements B in $SH^*(M)$ satisfying $[\Delta]B = 1$; the existence of dilations strongly constrains intersection properties of embedded Lagrangians [**S7**]. Theorem 5, or rather the variant discussed in Remark 6, implies *if there exists a dilation, e.g., an element* $x \in SH^1(M)$ *with* $[\Delta]x = 1$*, then on the Fukaya category of compact Lagrangians* $\mathcal{F}$*, there exists* $x' \in (\mathrm{HH}_{n+1}(\mathcal{F}))^\vee$ *with* $x' \circ [B] = [\overline{tr}]$, where $\overline{tr}$ is the canonical weak Calabi-Yau structure on the Fukaya category (see §1.1).

1.1. Calabi-Yau structures on the Fukaya category. Calabi-Yau structures are a type of cyclically symmetric duality structure on a dg or A_∞ category $\mathcal{C}$ generalizing the notion of a nowhere vanishing holomorphic volume form on a complex algebraic variety X in the case $\mathcal{C} = perf(X)$. As is well understood, there are two (in some sense dual) types of Calabi-Yau structures on A_∞ categories:

(1) *proper Calabi-Yau structures* [**KS1**] can be associated to *proper* categories $\mathcal{C}$ (those which have cohomologically finite-dimensional morphism spaces), abstract and refine the notion of integration against a nowhere vanishing holomorphic volume form. For $\mathcal{C} = perf(X)$ with

X a proper n-dimensional variety, the resulting structure in particular induces the Serre duality pairing with trivial canonical sheaf $\mathrm{Ext}^*(\mathcal{E}, \mathcal{F}) \times \mathrm{Ext}^*(\mathcal{F}, \mathcal{E}) \to \mathbf{k}[-n]$, Roughly, a proper Calabi-Yau structure on $\mathcal{C}$ (of dimension n) is a map $tr : \mathrm{HC}_*^+(\mathcal{C}) \to \mathbf{k}[-n]$ satisfying a non-degeneracy condition.

(2) *smooth Calabi-Yau structures* [**KV**] can be associated to *smooth* categories $\mathcal{C}$ (those with perfect diagonal bimodule), and abstract the notion of the nowhere vanishing holomorphic volume form itself, along with the induced identification (by contraction against the volume form) of polyvectorfields with differential forms. Roughly, a smooth Calabi-Yau structure on $\mathcal{C}$ (of dimension n) is a map $cotr : \mathbf{k}[n] \to \mathrm{HC}_*^-(\mathcal{C})$, or equivalently an element "$vol_{\mathcal{C}}$" in $\mathrm{HC}^-_{-n}(\mathcal{C})$, satisfying a non-degeneracy condition.

Precise definitions are reviewed in §6. When $\mathcal{C}$ is simultaneously smooth and proper, it is a folk result that the notions are equivalent; see [**GPS2**, Prop. 6.10].

In general, Calabi-Yau structures may not exist and when they do, there may be a non-trivial space of choices (see [**M3**] for an example). Calabi-Yau structures in either form induces non-trivial identifications between Hochschild invariants of the underlying category $\mathcal{C}$. [3] Moreover, categories with Calabi-Yau structures carry induced 2-dimensional TQFT operations on their Hochschild homology, associated to moduli spaces of Riemann surfaces with marked points [**C2**, **KS1**] (in the smooth case, this is still in progress but has been announced by Kontsevich-Vlassopoulous [**KV**]). In particular, Calabi-Yau structures play a central role in the mirror symmetry-motivated question of recovering Gromov-Witten invariants from the Fukaya category and to the related question of categorically recovering Hamiltonian Floer homology with all of its (possibly higher homotopical) operations. See [**C2**, **C3**, **K4**] for discussion and work around these questions and [**GPS2**] for applications to recovering genus-0 Gromov-Witten invariants.

Remark 8. A closely related to (1), and well studied, notion is that of a *cyclic A_∞ category*: this is an A_∞ category $\mathcal{C}$ equipped with a chain level perfect pairing

$$\langle -, - \rangle : \hom(X, Y) \times \hom(Y, X) \to \mathbf{k}[-n]$$

such that the induced correlation functions

$$\langle \mu^d(-, -, \dots, -), - \rangle$$

are strictly (graded) cyclically symmetric, for each d see e.g., [**C2**, **F2**, **CL**]. Although the property of being a cyclic A_∞ structure is not a homotopy invariant notion (i.e., not preserved under A_∞ quasi-equivalences), cyclic A_∞ categories and proper Calabi-Yau structures turn out to be weakly equivalent *in characteristic 0*, in the following sense. Any cyclic A_∞ category carries a canonical proper Calabi-Yau structure , and Kontsevich-Soibelman proved that a proper Calabi-Yau structure on any A_∞ category $\mathcal{C}$ determines a (essentially canonical) quasi-isomorphism between $\mathcal{C}$ and a cyclic A_∞ category $\widetilde{\mathcal{C}}$ [**KS1**, Thm. 10.7]. When $\mathrm{char}(\mathbf{k}) \neq 0$, these two notions differ in general, due to group cohomology obstructions to imposing cyclic symmetry. Moreoever, it seems that the notion of a proper Calabi-Yau structure is the "correct" one. (for instance, by Theorem 2, the compact Fukaya category always has one).

As a first application of Theorem 1, we verify the longstanding expectation that various compact Fukaya categories possess geometrically defined canonical Calabi-Yau structures:

Theorem 2. *The Fukaya category of compact Lagrangians has a canonical geometrically defined proper Calabi-Yau structure over any ground field* $\mathbf{k}$ *(over which the Fukaya category and* $\widetilde{\mathcal{OC}}$ *are defined).*

[3] In the proper case, there is an induced isomorphism between Hochschild cohomology and the linear dual of Hochschild homology. In the smooth case, there is an isomorphism between Hochschild cohomology and homology without taking duals.

In fact, this proper Calabi-Yau structure is easy to describe in terms of the cyclic open-closed map (c.f., Cor. 1): it is the composition of the map $\widetilde{\mathcal{OC}}^+ : \mathrm{HC}^+_*(\mathcal{F}) \to H^{*+n}(M, \partial M)((u))/uH^{*+n}(M, \partial M)[[u]]$[4] with the linear map to $\mathbf{k}$ which sends the top class $PD(pt) \cdot u^0 \in H^{2n}(M, \partial M)$ to 1, and all other generators $\alpha \cdot u^{-i}$ to 0. See §6 for more details.

As a consequence of the discussion in Remark 8, specifically [**KS1**, Thm. 10.7], we deduce that

Corollary 2. *If* $\mathrm{char}(\mathbf{k}) = 0$*, then the Fukaya category of compact Lagrangians carries, after quasi-isomorphism, a cyclic* A_∞ *structure.*

Remark 9. In the case of compact symplectic manifolds and over $\mathbf{k}$ = a Novikov field containing $\mathbb{R}$, Fukaya [**F2**] constructed a cyclic A_∞ model of the Floer cohomology algebra of a single compact Lagrangian, which was extended to multiple objects by Abouzaid-Fukaya-Oh-Ohta-Ono [**AFO**$^+$]

Remark 10. From the perspective of constructing 2d-TFTs from categories, Kontsevich-Soibelman [**KS2**] partly show (on the closed sector) that a proper Calabi-Yau structure can be used instead of the (weakly equivalent in characteristic 0) cyclic A_∞ structures considered in [**C2**]. One might similarly hope that, for applications of cyclic A_∞ structures to disc counting/open Gromov-Witten invariants developed in in [**F3**], a proper Calabi-Yau structure was in fact sufficient. See [**CL**] for related work.

Turning to smooth Calabi-Yau structures, in §6.2, we will establish the following existence of smooth Calabi-Yau structures, which applies to wrapped Fukaya categories of non-compact (Liouville) manifolds as well as Fukaya categories of compact manifolds:

Theorem 3. *Suppose our symplectic manifold M is* non-degenerate *in the sense of* [**G1**]*, meaning that the map* $[\mathcal{OC}] : \mathrm{HH}_{*-n}(\mathcal{F}) \to HF^*(M)$ *hits the unit* $1 \in HF^*(M)$*. Then, its (compact or wrapped) Fukaya category* $\mathcal{F}$ *possesses a canonical, geometrically defined* strong smooth Calabi-Yau structure.

Once more, the cyclic open-closed map gives an efficient description of this structure: it is the unique element $\mathrm{HC}^-_{-n}(\mathcal{F})$ mapping via $\widetilde{\mathcal{OC}}^-$ to the geometrically canonical lift $\tilde{1} \in H^*(CF^*(M)^{hS^1})$ of the unit $1 \in CF^*(M)$ described in §4.4. [5]

Remark 11. One can study other Fukaya categories of non-compact Lagrangians, such as *Fukaya categories of Landau-Ginzburg (LG) models* $(X, \pi : X \to \mathbb{C})$, and more generally *partially wrapped Fukaya categories* (or equivalently, wrapped Fukaya categories of *Liouville sectors*) . In contrast to compact Fukaya categories or wrapped Fukaya categories of Liouville manifolds, such categories are almost never Calabi-Yau in either sense, even if they are smooth or proper; indeed they typically arise as homological mirrors to perfect/coherent complexes on *non-Calabi-Yau* varieties.

These notions will be studied further in joint work with R. Cohen [**CG**], particularly with regards to the relationship of Calabi-Yau structures between the wrapped Fukaya category of a cotangent bundle and string topology category of the zero section.

1.2. Noncommutative Hodge-de-Rham degeneration for smooth and proper Fukaya categories. For a $C_{-*}(S^1)$ module $\mathcal{P}$, there is a canonical Tor spectral sequence converging to $H^*(\mathcal{P}_{hS^1})$ with first page $H^*(\mathcal{P}) \otimes_{\mathbf{k}} H^*(\mathbf{k}_{hS^1}) \cong H^*(\mathcal{P}) \otimes_{\mathbf{k}} H_*(\mathbb{CP}^\infty)$. When applied to the Hochschild complex $\mathcal{P} = \mathrm{CH}_*(\mathcal{C})$ of a (dg/A_∞) category $\mathcal{C}$, the resulting spectral sequence, from (many copies of) $\mathrm{HH}_*(\mathcal{C})$ to $\mathrm{HC}^+(\mathcal{C})$ is called the *Hochschild-to-cyclic* or *noncommutative Hodge-de-Rham (ncHDR) spectral sequence.* The latter name comes from the fact that when $\mathcal{C} = perf(X)$ is perfect complexes

[4]Recall that $C^*(M, \partial M)$ has the trivial $C_{-*}(S^1)$ module structure; the homology of the associated homotopy orbit complex is $H^{*+n}(M, \partial M)((u))/uH^{*+n}(M, \partial M)[[u]]$ where $|u| = 2$, as described in §2.

[5]As shown in [**G2**, **GPS2**], if $[\mathcal{OC}]$ hits 1, then $[\mathcal{OC}]$ is an isomorphism, and hence by Corollary 1, $[\widetilde{\mathcal{OC}}^-]$ is too. Hence one can speak about the unique element.

on a variety X, this spectral sequence is equivalent (via Hochschild-Kostant-Rosenberg (HKR) isomorphisms) to the usual Hodge-to-de-Rham spectral sequence from Hodge cohomology to de Rham cohomology

$$H^*(X, \Omega_X^*) \Rightarrow H^*_{dR}(X),$$

which degenerates in characteristic 0 whenever X is smooth and proper. Motivated by this, Kontsevich formulated the *noncommutative Hodge-de-Rham (ncHDR) degeneration conjecture* : for any smooth and proper category $\mathcal{C}$ in characteristic 0, its ncHDR spectral sequence degenerates. A general proof of this fact was recently given by Kaledin, following earlier work establishing it in the coconnective case.

Using the cyclic open-closed map, we can give a purely symplectic proof of the nc-HdR degeneration property for those $\mathcal{C}$ arising as Fukaya categories:

THEOREM 4. *Let $\mathcal{C} \subset \mathcal{F}(X)$ be a smooth and proper subcategory of the Fukaya category of a compact symplectic manifold over any field* $\mathbf{k}$ *(over which the Fukaya category is defined). Then, the nc Hodge-de-Rham spectral sequence for $\mathcal{C}$ degenerates.*

PROOF. The nc Hodge-de-Rham spectral sequence degenerates at page 1 if and only if $\mathcal{P}$ is isomorphic (in the category of $C_{-*}(S^1)$ modules) to a trivial $C_{-*}(S^1)$-module, e.g., if the $C_{-*}(S^1)$ action is trivializable. For compact symplectic manifolds M, recall that $CF^*(M) \cong H^*(M)$ has a canonically trivial(izable) $C_{-*}(S^1)$ action.

By earlier work [**GPS2**, **G4**], whenever $\mathcal{A}$ is smoooth, $\mathcal{OC}|_{\mathcal{A}}$ is an isomorphism from $\mathrm{HH}_{*-n}(\mathcal{A})$ onto a non-trivial summand S of $HF^*(M) \cong \mathrm{QH}^*(M)$; the $C_{-*}(S^1)$ action on this summand is trivial too. Theorem 1 shows that $\widetilde{\mathcal{OC}}|_{\mathcal{A}}$ induces an isomorphism in the category of $C_{-*}(S^1)$ modules between $\mathrm{CH}_{*-n}(\mathcal{A})$ and S with its trivial action, so we are done. $\square$

REMARK 12. Theorem 4 is true for a field $\mathbf{k}$ of any characteristic, at least whenever the Fukaya category and relevant structures are defined over $\mathbf{k}$ (for instance, in monotone settings). This is in stark contrast to the case of arbitrary smooth and proper dg or A_∞ categories in characteristic p, whose ncHDR spectral sequences need not degenerate. One explanation for this phenomenon is is that characteristic p Fukaya categories (whenever Lagrangians are monotone or tautologically unobstructed at least) seem to always admit a lift to second Witt vectors[6]

As is described in joint work (partly ongoing) with T. Perutz and N. Sheridan [**GPS2**, **GPS1**], the cyclic open-closed map $\widetilde{\mathcal{OC}}^-$ can further be shown to be a *morphism of semi-infinite Hodge structures*, a key step (along with the above degeneration property and construction of Calabi-Yau structure) in recovering Gromov-Witten invariants from the Fukaya category and enumerative mirror predictions from homological mirror theorems.

1.3. Outline of Paper. In §2, we recall a convenient model for the category of A_∞ modules over $C_{-*}(S^1)$ and various equivariant homology functors from this category. In §3 we review various Fukaya categories and the $C_{-*}(S^1)$ action on its (and indeed, any cohomologically unital A_∞ category's) *non-unital Hochschild chain complex*. In §4, we recall the construction of the A_∞ $C_{-*}(S^1)$ module structure on the (Hamiltonian) Floer chain complex, following [**BO**, **S4**] (note that our technical setup is slightly different, though equivalent). Then we prove our main results in §5. Some technical and conceptual variations on the construction of $\widetilde{\mathcal{OC}}$ (including Remark 6) are discussed at the end of this section, see §5.6. Finally, in §6 we apply our results to construct proper and smooth Calabi-Yau structures, proving Theorems 2 and 3.

Conventions. We work over a ground field $\mathbf{k}$ of arbitrary characteristic (though we note that all of our geometric constructions are valid over an arbitrary ring, e.g., $\mathbb{Z}$). All chain complexes will be graded *cohomologically*, including singular chains of any space, which hence have negative the homological grading and are denoted by $C_{-*}(X)$.

[6]The author wishes to thank Mohammed Abouzaid for discussions regarding this point.

Acknowledgements. I'd like to thank Paul Seidel for a very helpful conversation and Nick Sheridan for several helpful discussions about technical aspects of this paper such as signs. Part of this work was revised during a visit at the Institut Mittag-Leffler in 2015, who I'd like to thank for their hospitality.

2. Complexes with S^1 actions

In this section, we introduce a convenient model for the category of A_∞ $C_{-*}(S^1)$ modules in which the A_∞ $C_{-*}(S^1)$ can be described by a single (hierarchy) of maps satisfying equations. We also describe various equivariant homology complexes in this language in terms of simple formulae.

2.1. Definitions. Let $C_{-*}(S^1)$ denote the dg algebra of chains on the circle with coefficients in $\mathbf{k}$, graded cohomologically, with multiplication induced by the Pontryagin product $S^1 \times S^1 \to S^1$. This algebra is *formal*, or quasi-isomorphic to its homology, an exterior algebra on one generator Λ of degree -1 with no differential. Henceforth, by abuse of notation we take this exterior algebra as our working model for $C_{-*}(S^1)$

$$C_{-*}(S^1) := \mathbf{k}[\Lambda]/\Lambda^2, \quad |\Lambda| = -1, \tag{2.1}$$

and use the terminology $C_{-*}^{sing}(S^1)$ to refer to usual singular chains on S^1.

Definition 1. *A* strict-S^1 complex*, or a* chain complex with strict/dg S^1 action*, is a (unital) differential graded module over* $\mathbf{k}[\Lambda]/\Lambda^2$.

Let (M, d) be a strict-S^1 complex; by definition (M, d) is a co-chain complex (note: all complexes are graded cohomologically), and the dg $\mathbf{k}[\Lambda]/\Lambda^2$ module structure is equivalent to specificying the operation of multiplying by Λ

$$\Delta = \Lambda \cdot - : M_* \to M_{*-1}, \tag{2.2}$$

which must square to zero and anti-commute with d. In other words, (M, d, Δ) is what is known as a *mixed complex*, see e.g., [**B**, **K1**, **L2**].

We will need to work with the weaker notion of an A_∞ action, or rather an A_∞ module structure over $C_{-*}(S^1) = \mathbf{k}[\Lambda]/\Lambda^2$. Recall that a *(left)* A_∞ module M [**K3**, **S5**, **S3**, **G2**] over the associative graded algebra $A = \mathbf{k}[\Lambda]/\Lambda^2$ is a graded $\mathbf{k}$-module M equipped with maps

$$\mu^{k|1} : A^{\otimes k} \otimes M \to M, \quad k \geq 0 \tag{2.3}$$

of degree $1 - k$, satisfying the A_∞ module equations described in [**S3**] or [**G2**, (2.35)]. Since $A = \mathbf{k}[\Lambda]/\Lambda^2$ is unital, we can work with modules that are also *strictly unital* (see [**S3**, (2.6)]); this implies that all multiplications by a sequence with at least one unit elements is completely specified,[7] and hence the only non-trivial structure maps to define are the operators

$$\delta_k := \mu_M^{k|1}(\underbrace{\Lambda, \ldots, \Lambda}_{k \text{ copies}}, -) : M \to M[1-2k], \ k \geq 0. \tag{2.4}$$

The A_∞ module equations are equivalent to the following relations for (2.4) for each $s \geq 0$,

$$\sum_{i=0}^{s} \delta_i \delta_{s-i} = 0. \tag{2.5}$$

We summarize the discussion so far with the following definition:

Definition 2. *An* S^1-complex*, or a* chain complex with a A_∞ S^1 action*, is a stricly unital (right)* A_∞ *module* M *over* $\mathbf{k}[\Lambda]/\Lambda^2$*. Equivalently, it is a graded* $\mathbf{k}$*-module* M *equipped with operations* $\{\delta_k : M \to M[1-2k]\}_{k\geq 0}$ *satisfying, for each* $s \geq 0$*, the hierarchy of equations* (2.5).

[7] More precisely $\mu^{1|1}(1, \mathbf{m}) = \mathbf{m}$ and $\mu^{k|1}(\ldots, 1, \ldots, \mathbf{m}) = 0$ for $k > 1$.

REMARK 13. If X is a space with S^1 action, then as described in §22, $C_{-*}(X)$ carries a dg $C_{-*}^{sing}(S^1)$ module structure. Under the A_∞ equivalence $C_{-*}^{sing}(S^1) \cong \mathbf{k}[\Lambda]/\Lambda^2$, it follows that $C_{-*}(X)$ carries an A_∞ (not necessarily dg!) $\mathbf{k}[\Lambda]/\Lambda^2$ module structure, which can be made strictly unital (by [**L1**, Thm. 3.3.1.2] or by passing to normalized chains). By strictification, one can recover a dg $\mathbf{k}[\Lambda]/\Lambda^2$ module equivalent (as A_∞ modules) to $C_{-*}(X)$; or by passing to a further quotient complex of *unordered singular chains*; see e.g., [**CG**, Appendix.].

REMARK 14. There are multiple sign conventions for A_∞ modules; most notably the two most common conventions appearing in the A_∞ algebra/category and A_∞ module equations appear in [**S3**, (2.6)] and [**S5**, (1j)] respectively. However, the differences in sign conventions are completely irrelevant for strictly unital $A = \mathbf{k}[\Lambda]/\Lambda^2$ modules, as the relevant signs all vanish (they involve sums of reduced degrees of some elements in $\bar{A} = \mathrm{span}_{\mathbf{k}}(\Lambda)$, but these degrees are all zero).

For $s = 0$, (2.5) says simply that the differential $d = \delta_0$ squares to 0; for $s = 1$, (2.5) implies $\delta := \delta_1$ anti-commutes with d, and for $s = 2$, $(\delta)^2 = -(d\delta_2 + \delta_2 d)$, or that δ^2 is chain-homotopic to zero, but not strictly zero, as measured by the chain homotopy δ_2.

REMARK 15. The data $(M, \delta_k, k \geq 0)$ is sometimes referred to as an ∞-*mixed complex* or an S^1-*complex* (c.f., [**BO**, **Z**] but note the alternate homological grading conventions in the first reference).

S^1-complexes, as strictly unital A_∞ modules over the augmented algebra $A = \mathbf{k}[\Lambda]/\Lambda^2$, are the objects of a dg category *dg* category which we will call

$$S^1\text{−mod} := uA\text{−mod} \tag{2.6}$$

(compare [**S3**, p. 90, 94] for the definition of $mod(A) = mod(A, \mathbf{k})$) whose morphisms and compositions we now recall. Denote by $\epsilon : A \to \mathbf{k}$ the augmentation map, and $\bar{A} = \ker \epsilon = \mathrm{span}_{\mathbf{k}}(\Lambda)$ the augmentation ideal. Let M and N be two strictly unital A_∞ A-modules. A *unital pre-morphism of degree* k from M to N of degree k is a collection of maps $F^{d|1} : \overline{A}^{\otimes d} \otimes M \to N$, $d \geq 0$, of degree $k - d$, or equivalently since $\dim_{\mathbf{k}}(\overline{A}) = 1$ in degree -1, a collection of operators

$$\begin{aligned} F &= \{F^d\}_{d \geq 0} \\ F^d &:= F^{d|1}(\underbrace{\Lambda, \ldots, \Lambda}_{d \text{ copies}}, -) : M \to N[k - 2d]. \end{aligned} \tag{2.7}$$

The space of pre-morphisms of each degree form the graded space of morphisms in S^1−mod, which we will denote by $\mathrm{Rhom}_{S^1}(-,-)$:

$$\begin{aligned} \mathrm{Rhom}_{S^1}(M, N) &:= \bigoplus_{k \in \mathbb{Z}} \mathrm{Rhom}^k_{S^1}(M, N) := \oplus_k \hom_{grVect}(T(\overline{A}[1]) \otimes M, N[k]) \\ &= (\bigoplus_{k \in \mathbb{Z}} \hom_{grVect}(\oplus_{d \geq 0} M[2d], N[k])). \end{aligned} \tag{2.8}$$

There is a differential ∂ on (2.8) described in [**S3**, p. 90]; in terms of the simplified form of pre-morphisms (2.7), one has

$$(\partial F)^s = \sum_{i=0}^{s} F^i \circ \delta^M_{s-i} - (-1)^{\deg(F)} \sum_{j=0}^{s} \delta^N_{s-j} \circ F^j. \tag{2.9}$$

An A_∞ $\mathbf{k}[\Lambda]/\Lambda^2$ *module homomorphism*, or S^1-*complex homomorphism* is a pre-morphism $F = \{F^d\}$ which is closed, e.g., $\partial F = 0$. In particular, F is an A_∞ module homomorphism if the following equations are satisfied, for each s:

$$\sum_{i=0}^{s} F^i \circ \delta^M_{s-i} = (-1)^{\deg(F)} \sum_{j=0}^{s} \delta^N_{s-j} \circ F^j. \tag{2.10}$$

Note that the $s = 0$ equation reads $F^0 \circ \delta_0^M = (-1)^{\deg(F)}\delta_0^N \circ F^0$, so (if $\partial F = 0$) F^0 induces a cohomology level map $[F^0] : H^*(M) \to H^{*+\deg(F)}(N)$. A module homomorphism (or closed morphism) F is said to be a *quasi-isomorphism* if $[F^0]$ is an isomorphism on cohomology.

Remark 16. There is an enlarged notion of a *non-unital* pre-morphism (used for modules which aren't necessarily strictly unital), which is a collection of maps $F^d : A^{\otimes d} \otimes M \to N$ instead of $\overline{A}^{\otimes d} \otimes M \to N$. Any pre-morphism as we've defined induces a non-unital pre-morphism by declaring $F^d(\dots, 1, \dots, \mathbf{m}) = 0$. For strictly unital modules, the resulting inclusion is a quasi-isomorphism.

Remark 17. When M and N are *dg* modules, or strict S^1 complexes, $\mathrm{Rhom}_{S^1}(M, N)$ is a *reduced bar model* of the chain complex of derived $\mathbf{k}[\Lambda]/\Lambda^2$ module homomorphisms, which is one of the reasons we've adopted the terminology "Rhom". In the A_∞ setting, we recall that there is no sensible "non-derived" notion of a $\mathbf{k}[\Lambda]/\Lambda^2$ module map (compare [**S3**]).

The composition in the category $S^1-\mathrm{mod}$

$$\mathrm{Rhom}_{S^1}(N, P) \otimes \mathrm{Rhom}_{S^1}(M, N) \to \mathrm{Rhom}_{S^1}(M, P) \tag{2.11}$$

is defined by

$$(G \circ F)^s = \sum_{j=0}^{s} G^{s-j} \circ F^j \tag{2.12}$$

Remark 18. If M is any S^1-complex, then its endomorphisms $\mathrm{Rhom}_{S^1}(M, M)$ equipped with composition, form a dg algebra. As an example, consider $M = \mathbf{k}$, with trivial module structure (determined by the augmentation $\epsilon : \mathbf{k}[\Lambda]/\Lambda^2 \to \mathbf{k}$). It is straightforward to compute that, as a dga

$$\mathrm{Rhom}_{S^1}(\mathbf{k}, \mathbf{k}) \cong \mathbf{k}[u],\ |u| = 2. \tag{2.13}$$

(in terms of the definition of morphism spaces (2.8), u corresponds to the unique morphism $G = \{G^d\}_{d \geq 0}$ of degree $+2$ with $G^1 = \mathrm{id}$ and $G^s = 0$ for $s \neq 1$).

In addition to taking the morphism spaces, one can define the (derived) *tensor product* of S^1-complexes N and M: using the isomorphism $A \cong A^{op}$ coming from commutativity of $A = \mathbf{k}[\Lambda]/\Lambda^2$, first view N as a *right* A_∞ A *module* (see [**S3**, p. 90, 94] where the category of right A modules are called *mod*$(\mathbf{k}, A)$, [**S5**, (1j)], [**G2**, §2]) and then take the usual tensor product of N and M over A (see [**S3**, p. 91] or [**G2**, §2.5]). The resulting chain complex has underling graded vector space

$$\begin{aligned} N \otimes^{\mathbb{L}}_{S^1} M &:= \oplus_{d \geq 0} N \otimes \overline{A}[1]^{\otimes d} \otimes M \\ &= \bigoplus_{d \geq 0} (N \otimes_{\mathbf{k}} M)[2d] \end{aligned} \tag{2.14}$$

(the degree s part is $\bigoplus_{d\geq 0} \bigoplus_s N_t \otimes M_{s+2d-t}$). Let us refer to an element $n \otimes m$ of the dth summand of this complex by suggestive notation $n \otimes \underbrace{\Lambda \otimes \cdots \otimes \Lambda}_{d \text{ times}} \otimes m$ as in the first line of (2.14). With this notation, the differential on (2.14) acts as

$$\partial(n \otimes \underbrace{\Lambda \otimes \cdots \otimes \Lambda}_{d} \otimes m) = \sum_{i=0}^{d} \left((-1)^{|m|} \delta_i^N n \otimes \underbrace{\Lambda \otimes \cdots \otimes \Lambda}_{d-i} \otimes m + n \otimes \underbrace{\Lambda \otimes \cdots \otimes \Lambda}_{d-i} \otimes \delta_i^M m \right). \tag{2.15}$$

(here our sign convention follows [**G2**, §2.5] rather than [**S3**], though the sign difference is minimal).

Remark 19. Analogously to Remark 17, if M and N are dg $\mathbf{k}[\Lambda]/\Lambda^2$-modules, the chain complex described above computes the ordinary derived tensor product, whose homology is $\mathrm{Tor}_{\mathbf{k}[\Lambda]/\Lambda^2}(M, N)$. While we have therefore opted for the notation $N \otimes^{\mathbb{L}}_A M$ in this more general setting, this (derived) tensor product is typically written in the A_∞ literature simply as $N \otimes_A M$.

The pairing (2.14) is suitably functorial with respect to morphisms of the S^1-complexes involved, meaning that $-\otimes_{S^1} N$ and $M \otimes_{S^1} -$ both induce dg functors from S^1-mod to chain complexes (compare [**S3**, p. 92]). For instance, if $F = \{F^j\} : M_0 \to M_1$ is a pre-morphism of S^1-complexes, then there are induced maps

$$
\begin{aligned}
&F_\sharp : N \otimes^{\mathbb{L}}_{S^1} M_0 \to N \otimes^{\mathbb{L}}_{S^1} M_1 \qquad (2.16)\\
&n \otimes \underbrace{\Lambda \otimes \cdots \otimes \Lambda}_{d} \otimes m \mapsto \sum_{j=0}^{d} n \otimes \underbrace{\Lambda \otimes \cdots \otimes \Lambda}_{d-j} \otimes F^j(m);\\
&F_\sharp : M_0 \otimes^{\mathbb{L}}_{S^1} N \to M_1 \otimes^{\mathbb{L}}_{S^1} N \qquad (2.17)\\
&m \otimes \underbrace{\Lambda \otimes \cdots \otimes \Lambda}_{d} \otimes n \mapsto \sum_{j=0}^{d} (-1)^{\deg(F)\cdot|n|} F^j(m) \otimes \underbrace{\Lambda \otimes \cdots \otimes \Lambda}_{d-j} \otimes n;
\end{aligned}
$$

which are chain maps if $\partial(F) = 0$.

Hom and tensor complexes of S^1-complexes, as in any category of A_∞ modules, satisfy the following strong homotopy invariance properties:

Proposition 1 (Homotopy invariance). *If $F : M \to M'$ is any quasi-isomorphism of S^1-complexes (meaning $\partial(F) = 0$ and $[F^0] : H^*(M) \xrightarrow{\cong} H^*(M')$ is an isomorphism), then composition with F induces quasi-isomorphisms of hom and tensor complexes:*

$$
\begin{aligned}
F \circ \cdot : \mathrm{Rhom}_{S^1}(M', P) &\xrightarrow{\sim} \mathrm{Rhom}_{S^1}(M, P)\\
\cdot \circ F : \mathrm{Rhom}_{S^1}(P, M) &\xrightarrow{\sim} \mathrm{Rhom}_{S^1}(P, M')\\
F_\sharp : N \otimes^{\mathbb{L}}_{S^1} M &\xrightarrow{\sim} N \otimes^{\mathbb{L}}_{S^1} M'.\\
F_\sharp : M \otimes^{\mathbb{L}}_{S^1} N &\xrightarrow{\sim} M' \otimes^{\mathbb{L}}_{S^1} N.
\end{aligned} \tag{2.18}
$$

The proof is a standard argument involving filtration of the above hom and tensor complexes by "length" (with respect to the number of $\overline{A}^{\otimes d}$ factors).

Let $(P, \{\delta_i^P\})$ and $(Q, \{\delta_j^Q\}_j)$ be S^1-complexes, and $f : P \to Q$ a chain map of some degree $\deg(f)$ (with respect to the δ_0^P and δ_0^Q differentials). An S^1*-equivariant enhancement of* f is a degree $\deg(f)$ homomorphism $\mathbf{F} = \{\mathbf{F}^i\}_{i\geq 0}$ of S^1-complexes (e.g., a closed morphism, so $\mathbf{F}$ satisfies (2.10)) with $[\mathbf{F}^0] = [f]$.

Remark 20. Note that there are a series of obstructions to the existence of an S^1-equivariant enhancement of a given chain map f; for instance a necessary condition is the vanishing of $[f] \circ [\delta_1^P] - [\delta_1^Q] \circ [f] = 0$.

Finally, we note that, just as the product of S^1 spaces $X \times Y$ possesses a diagonal action, the (linear) tensor product of S^1-complexes is again an S^1 complex.

Lemma 1. *If $(M, \delta^M_{eq} = \sum_{i=0}^\infty \delta_j^M u^j)$ and $(N, \delta^N_{eq} = \sum_{i=0}^\infty \delta_i^N u^i)$ are S^1-complexes, then the graded vector space $M \otimes N$ is naturally an S^1-complex with $\delta^{M\otimes N}_{eq} = \sum_{i=0}^\infty \delta_k^{M\otimes N} u^k$, where*

$$\delta_k^{M\otimes N}(\mathbf{m}\otimes\mathbf{n}) := (-1)^{|\mathbf{n}|}\delta_k^M \mathbf{m}\otimes\mathbf{n} + \mathbf{m}\otimes\delta_k^N\mathbf{n} \tag{2.19}$$

We call the resulting S^1 action on $M \otimes N$ the diagonal S^1-action.

Proof. We compute

$$\delta_j^{M\otimes N}\delta_k^{M\otimes N}(\mathbf{m}\otimes\mathbf{n}) = \delta_j^M\delta_k^M\mathbf{m}\otimes\mathbf{n} + (-1)^{|\mathbf{n}|+1}\delta_j^M\mathbf{m}\otimes\delta_k^N\mathbf{n} + (-1)^{|\mathbf{n}|}\delta_k^M\mathbf{m}\otimes\delta_j^N\mathbf{n} + \mathbf{m}\otimes\delta_j^N\delta_k^N\mathbf{n} \tag{2.20}$$

Summing over all $j + k = s$, the middle two terms cancel in pairs and the sums of the leftmost terms (respectively rightmost) terms respectively vanish because M (respectively N) is an S^1-complex. □

2.2. Equivariant homology groups. Let M be an S^1 complex. Let $\mathbf{k}$ denote the strict trivial S^1 complex concentrated in degree 0, induced by the augmentation map $\epsilon : \mathbf{k}[\Lambda]/\Lambda^2 \to \mathbf{k}$.

Definition 3. *The* homotopy orbit complex *of M is the (derived) tensor product of M with $\mathbf{k}$ over $C_{-*}(S^1)$:*

$$M_{hS^1} := \mathbf{k} \otimes^{\mathbb{L}}_{S^1} M. \tag{2.21}$$

Remark 21. When $M = C_{-*}(X)$, with S^1-complex induced by a topological S^1 action on X as in Remark 13, the complex (2.21) computes the Borel equivariant homology of X:

$$M_{hS^1} \simeq C^{S^1}_{-*}(X) = C_{-*}(X \times_{S^1} ES^1) \cong C_{-*}(X_{hS^1}).$$

In some sense, this is a justification for the usage of the the $_{hS^1}$ notation.

Remark 22. The topologically minded reader should recall how to compute the (Borel) equivariant homology a space X with $G = S^1$ action via chain level data, as follows: First, note $C_{-*}(G)$ is a dg algebra (with Pontryagin product induced by $G \times G \to G$), and $C_{-*}(EG)$ and $C_{-*}(X)$ are dg modules over $C_{-*}(G)$ (with Pontryagin multiplication again). Then, one recalls that the equivariant homology chain complex, which is typically described as the singular chains on the *homotopy orbit space* $X_{hG} \simeq X \times_G EG$, can be computed as:

$$C^G_{-*}(X) := C_{-*}(X \times_G EG) \simeq C_{-*}(X) \otimes_{C_{-*}G} C_{-*}(EG). \tag{2.22}$$

(see e.g., [**M2**, Thm. 7.27]) On the other hand, the morphism of $C_{-*}(G)$ modules $C_{-*}(EG) \to \mathbf{k} \cong C_{-*}(pt)$ realizes $C_{-*}(EG)$ as a *free resolution* of $\mathbf{k}$ (as G acts on EG freely), hence (2.22) computes the *derived tensor product* of $C_{-*}(X)$ with $\mathbf{k}$, and we have

$$H^G_{-*}(X) \cong H_{-*}(C_{-*}X \otimes^{\mathbb{L}}_{C_{-*}G} \mathbf{k}) \tag{2.23}$$

This justifies the use of the terminology $C_{-*}(X)_{hG} \cong C_{-*}X \otimes^{\mathbb{L}}_{C_{-*}G} \mathbf{k}$.

Definition 4. *The* homotopy fixed point complex *of M is the chain complex of morphisms from $\mathbf{k}$ to M in the category of S^1-complexes:*

$$M^{hS^1} := \mathrm{Rhom}_{S^1}(\mathbf{k}, M). \tag{2.24}$$

Remark 23. To motivate the usage "homotopy fixed points," note that in the topological category, the usual fixed points of a G action can be described as $Maps_G(pt, X)$. Note that when $M = C_{-*}(X)$, for X a space with S^1 action, it is not necessarily true (unlike the case of homotopy orbits discussed in remark 21) that $C_{-*}(X^{hS^1}) = C_{-*}(Maps_{S^1}(ES^1, X))$ is equal to $(C_{-*}(X))^{hS^1}$. There is, however, always a map $C_{-*}(X^{hS^1}) \to (C_{-*}(X))^{hS^1}$.

Remark 24. Composition induces a natural action of

$$\mathrm{Rhom}_{S^1}(\mathbf{k}, \mathbf{k}) = \mathbf{k}[u] \quad (|u| = 2) = H^*(BS^1) \tag{2.25}$$

on the homotopy fixed point complex. There is a third important equivariant homology complex, called the *periodic cyclic*, or *Tate* complex of M, defined as the localization of M^{hS^1} away from $u = 0$;

$$M^{Tate} := M^{hS^1} \otimes_{\mathbf{k}[u]} \mathbf{k}[u, u^{-1}]. \tag{2.26}$$

The Tate construction sits in an exact sequence between the homotopy orbits and fixed points.

Remark 25 (Gysin sequences). It is straightforward from the viewpoint of A_∞ $C_{-*}(S^1)$ modules to explain the appearance of various Gysin and periodicity sequences. Take for instance the *Gysin exact triangle*

$$M_{hS1} \to M_{hS1}[2] \to M \stackrel{[1]}{\to} \cdots$$

This is a manifestation of a canonical exact triangle of objects in S^1-mod:

$$\mathbf{k} \stackrel{u}{\to} \mathbf{k}[2] \to \mathbf{k}[\Lambda]/\Lambda^2 \stackrel{[1]}{\to} \cdots$$

(recall in Remark 18 it was shown $\mathrm{Rhom}_{S^1}(\mathbf{k},\mathbf{k}) \cong \mathbf{k}[u]$), pushed forward by the functor $\mathrm{Rhom}_{S^1}(\cdot, M)$. The others exact sequences arise similarly.

As a special case of the general homotopy-invariance properties of A_∞ modules stated in Proposition 1, we have:

COROLLARY 3. *If $F : M \to N$ is a homomorphism of S^1-complexes (meaning a closed morphism), it induces chain maps between equivariant theories*

$$F^{hS^1} : M^{hS^1} \to N^{hS^1} \tag{2.27}$$

$$F_{hS^1} : M_{hS^1} \to N_{hS^1} \tag{2.28}$$

$$F^{Tate} : M_{Tate} \to N_{Tate} \tag{2.29}$$

If F is a quasi-isomorphism of S^1-complexes (meaning simply $[F^0]$ is a homology isomorphism), then (2.27)-(2.29) *are quasi-isomorphisms of chain complexes.* □

Functoriality further tautologically implies that

PROPOSITION 2. *If $F : M \to N$ is a homomorphism of S^1-complexes, then the various induced maps* (2.27) - (2.29) *intertwine all of the long exact sequences for (equivariant homology groups of) M with those for N.* □

2.3. u-linear models for S^1-complexes. It is convenient to package the data described in the previous two sections into "u-linear generating functions", in the following way: Let u be a formal variable of degree $+2$. Let us use the abuse of notation

$$M[[u]] := M \widehat{\otimes}_{\mathbf{k}} \mathbf{k}[u]$$

for the u-adically completed tensor product in the category of graded vector spaces; in other words $M[[u]] := \oplus_k M[[u]]_k$, where $M[[u]]_k = \{\sum_{i=0}^{\infty} m_i u^i \mid m_i \in M_{k-2i}\}$. Then, we frequently write an S^1-complex $(M, \{\delta_k\}_{k\geq 0})$ as a $\mathbf{k}$-module M equipped with a map

$$\delta_{eq}^M = \sum_{i=0}^{\infty} \delta_i^M u^i : M \to M[[u]] \tag{2.30}$$

of total degree 1, satisfying $\delta_{eq}^2 = 0$ (where we are implicitly conflating δ_{eq} with its u-linear extension to a map $M[[u]] \to M[[u]]$ in order to u-linearly compose and obtain a map $M \to M[[u]]$).

Pre-morphisms from M to N of degree k can similarly be recast as maps $F_{eq} = \sum_{i=0}^{\infty} F_i u^i : M \to N[[u]]$ of pure degree k (so each F_i has degree $k - 2i$). The differential on pre-morphisms can be described u-linearly as

$$\partial(F_{eq}) = F_{eq} \circ \delta_{eq}^M - \delta_{eq}^N \circ F_{eq}, \tag{2.31}$$

and composition is simply the u-linear composition $G_{eq} \circ F_{eq}$ (again, one implicitly u-linearly extends G_{eq} and then u-linearly composes); explicitly $(\sum_{i\geq 0} G_i u^i) \circ (\sum_{j\geq 0} F_j u^j) = \sum_{k\geq 0} (\sum_{i+j=k} G^i \circ F^j) u^k$.

With respect to this packaging, the formulae for various equivariant homology groups can be given the following, more readable form:

$$M_{hS^1} = (M((u))/uM[[u]],\ \delta_{eq}) \tag{2.32}$$

$$M^{hS^1} = (M[[u]],\ \delta_{eq}) \tag{2.33}$$

$$M^{Tate} = (M((u)), \delta_{eq}) \tag{2.34}$$

where again, we use the abuse of notation $M((u)) = M[[u]] \otimes_{\mathbf{k}[u]} \mathbf{k}[u, u^{-1}]$ (on the other hand, note that (2.32) is *not* completed). As before, any homomorphism (that is, closed morphism) of S^1 complexes $F_{eq} = \sum_{i=0}^{\infty} F^i u^i$ induces a $\mathbf{k}[u]$-linear chain map between homotopy-fixed point complexes by u-linearly extended composition, and hence, by tensoring over $\mathbf{k}[u]$ with $\mathbf{k}((u))/u\mathbf{k}[[u]]$ or $\mathbf{k}((u))$, chain maps between homotopy orbit and Tate complex constructions.

Remark 26. This u-linear lossless packaging of the data describing an S^1-complex is a manifestation of *Koszul duality*; in the case of $A = \mathbf{k}[\Lambda]/\Lambda^2$, it posits that there is a fully faithful embedding, $\mathrm{Rhom}(\mathbf{k}, -) = (-)^{hS^1}$ from A-modules into $B := \mathrm{Rhom}_A(\mathbf{k}, \mathbf{k}) = \mathbf{k}[u]$ modules.

From the u-linear point of view, it is easy to observe that the exact triangle of $\mathbf{k}[u]$ modules $\mathbf{k}[[u]] \to \mathbf{k}((u)) \to \mathbf{k}((u))/u\mathbf{k}[[u]]$ induces a (functorial in M) exact triangle between equivariant homology chain complexes $M^{hS^1} \to M^{Tate} \to M_{hS^1}[2] \xrightarrow{[1]}$.

3. Circle action on the open sector

Recall that an A_∞ *category* $\mathcal{C}$ consists of the data of

- a collection of objects ob $\mathcal{C}$
- for each pair of objects X, X', a graded vector space $\hom_{\mathcal{C}}(X, X')$
- for any set of $d+1$ objects $X_0, \ldots, X_d$, higher composition maps

$$\mu^d : \hom_{\mathcal{C}}(X_{d-1}, X_d) \otimes \cdots \otimes \hom_{\mathcal{C}}(X_0, X_1) \to \hom_{\mathcal{C}}(X_0, X_d) \tag{3.1}$$

of degree $2-d$, satisfying the (quadratic) A_∞ relations, for each $k > 0$:

$$\sum_{i,l} (-1)^{\maltese_i} \mu_{\mathcal{C}}^{k-l+1}(x_k, \ldots, x_{i+l+1}, \mu^l_{\mathcal{C}}(x_{i+l}, \ldots, x_{i+1}), x_i, \ldots, x_1) = 0. \tag{3.2}$$

with sign

$$\maltese_i := ||x_1|| + \cdots + ||x_i||. \tag{3.3}$$

where $|x|$ denotes degree and $||x|| := |x| - 1$ denotes reduced degree.

The first two equations ($k = 1, 2$) of (3.2) says in particular that $(\mu^1)^2 = 0$, so μ^1 is a differential, and the cohomology level maps $[\mu^2]$ are a genuine composition for the (non-unital) category with morphisms

$$\mathrm{Hom}_{H^*(\mathcal{C})}(X, Y) := H^*(\hom_{\mathcal{C}}(X, Y), \mu^1) \tag{3.4}$$

We will implicitly always assume that $\mathcal{C}$ is *cohomologically unital*, meaning the cohomology level morphism spaces (3.4) have identity morphisms, making $H^*(\mathcal{C})$ a genuine category.

For any (cohomologically unital) A_∞ category $\mathcal{C}$, a certain chain complex computing Hochschild homology called the *non-unital Hochschild complex* $\mathrm{CH}^{nu}_*(\mathcal{C})$ possesses the structure of a strict S^1-complex (in the sense of Definition 1), which is an invariant (up to quasi-isomorphism) of the quasi-equivalence class (and indeed Morita equivalence class) of $\mathcal{C}$. Whenever $\mathcal{C}$ is strictly unital, the relevant S^1-structure is known to be compatible with the traditional S^1-structure defined on the Hochschild complex by Connes, Tsygan, and Loday-Quillen [**C1**, **T2**, **LQ**].

Turning to our geometric context, we review the construction of the A_∞ structure on the (wrapped or compact) Fukaya category. Fukaya categories are well known to be cohomologically unital, and hence it follows that the non-unital Hochschild complex of the Fukaya category (computes Hochschild homology and) carries a canonical up to quasi-isomorphism strict S^1 action.

3.1. Hochschild and cyclic homology.

Let $\mathcal{C}$ be an A_∞ category. The *Hochschild*, or *cyclic bar complex* of $\mathcal{C}$ is the direct sum of all cyclically composable sequences of morphism spaces in $\mathcal{C}$:

$$\mathrm{CH}_*(\mathcal{C}) := \bigoplus_{k \geq 0, X_{i_0}, \ldots, X_{i_k} \in \mathrm{ob}\ \mathcal{C}} \hom_{\mathcal{C}}(X_{i_k}, X_{i_0}) \otimes \hom_{\mathcal{C}}(X_{i_{k-1}}, X_{i_k}) \otimes \cdots \otimes \hom_{\mathcal{C}}(X_{i_0}, X_{i_1}). \tag{3.5}$$

Remark 27. Frequently the notation $\mathrm{CH}_*(\mathcal{C}, \mathcal{C})$ is used to emphasize that Hochschild homology is taken here with *diagonal coefficients*, rather than coefficients in another bimodule. We will adopt the slightly simpler $\mathrm{CH}_*(\mathcal{C})$.

The differential b acts on Hochschild chains by summing over ways to cyclically collapse elements by any of the A_∞ structure maps:

$$
\begin{aligned}
b(\mathbf{x}_d \otimes x_{d-1} \otimes \cdots \otimes x_1) = & \\
& \sum (-1)^{\#_k^d} \mu^{d-i}(x_k, \ldots, x_1, \mathbf{x}_d, x_{d-1}, \ldots, x_{k+i+1}) \otimes x_{k+i} \otimes \cdots \otimes x_{k+1} \\
& + \sum (-1)^{\maltese_1^s} \mathbf{x}_d \otimes \cdots \otimes \mu^j(x_{s+j+1} \otimes \cdots \otimes x_{s+1}) \otimes x_s \otimes \cdots \otimes x_1.
\end{aligned} \tag{3.6}
$$

with signs

$$
\maltese_i^k := \sum_{j=i}^{k} ||x_i|| \tag{3.7}
$$

$$
\#_k^d := \maltese_1^k \cdot (1 + \maltese_{k+1}^d) + \maltese_{k+1}^{d-1} + 1 \tag{3.8}
$$

In this complex, Hochschild chains are graded as follows:

$$
\deg(\mathbf{x}_d \otimes x_{d-1} \otimes \cdots \otimes x_1) := \deg(\mathbf{x}_d) + \sum_{i=1}^{d-1} \deg(x_i) - d + 1 = |\mathbf{x}_d| + \sum_{i=1}^{d-1} ||x_i||. \tag{3.9}
$$

If $\mathcal{C}$ is a *strictly unital* A_∞ category, meaning that one has chain level identity elements $e_X^+ \in \hom_{\mathcal{C}}(X, X)$, for all $X \in \mathrm{ob}\ \mathcal{C}$, satisfying

$$
\begin{gathered}
\mu^1(e_X^+) = 0 \\
(-1)^{|y|}\mu^2(e_{X_0}^+, y_0) = y_0 = \mu^2(y_0, e_{X_1}^+) \\
\mu^d(\cdots, e_X^+, \cdots) = 0 \text{ for } d > 2.
\end{gathered} \tag{3.10}
$$

Connes [**C1**, §II] observed that via combining the discrete $\mathbb{Z}/k\mathbb{Z}$ cyclic rotation operation and insertions of e_X^+, one can equip the Hochschild chain complex with a strict S^1 action. The relevant operator B, known as the *Connes' B operator*, is defined as the composition of several basic operations.

First, denote by t the (signed) cyclic permutation operator, generating the $\mathbb{Z}/k\mathbb{Z}$ cyclic action

$$
t : x_k \otimes \cdots \otimes x_1 \mapsto (-1)^{||x_1|| \cdot \maltese_2^k + ||x_1|| + ||x_k||} x_1 \otimes x_k \otimes \cdots \otimes x_2, \tag{3.11}
$$

Let N denote the *norm* of this operation; that is the sum of all powers of t (this depends on k, the length of a given Hochschild chain):

$$
N : x_k \otimes \cdots \otimes x_1 = \sigma \mapsto (1 + t + t^2 + \cdots + t^{k-1})\sigma \tag{3.12}
$$

Let s denote the operation of inserting a strict unit e_X^+ (for each Hochschild chain there is only one X which ensures the resulting sequence of morphisms remains cyclically composable).

$$
\begin{aligned}
s : x_k \otimes \cdots \otimes x_1 \mapsto (-1)^{||x_k|| + \maltese_1^k + 1} e_{X_{i_k}}^+ \otimes x_k \otimes \cdots \otimes x_1, & \\
\text{where } x_k \in \hom_{\mathcal{C}}(X_{i_k}, X_{i_0}) &
\end{aligned} \tag{3.13}
$$

(recall that our sign convention involves Koszul signs where every operation by default acts on the right). The Connes' B operator is defined as

$$
B := (1 - t)sN. \tag{3.14}
$$

The proof of the following is identical to the proof for associative algebras:

Lemma 2 (Compare [**L2**] §2.1). *If $\mathcal{C}$ is strictly unital, then $B^2 = 0$ and $Bb + bB = 0$ on the chain level. In other words, $(CH_*(\mathcal{C}), \delta_0 = b, \delta_1 = B, \delta_{\geq 2} = 0)$ is a complex with strict S^1 action (e.g., a* mixed complex*).* □

There is also a quasi-isomorphic *normalized Hochschild complex* $CH_*^{red}(\mathcal{C}, \mathcal{C})$, a quotient of $\mathrm{CH}_*(\mathcal{C}, \mathcal{C})$ in which at most the first element of a Hochschild chain is allowed to be a strict unit. On the reduced complex, Connes' B has the simpler form:

$$B^{red} = sN.$$

REMARK 28. Following [**S5**, **A**, **G2**], we have adopted the following Koszul sign convention for A_∞ algebra and Hochschild complexes: we should think of a Hochschild chain $x_k \otimes \cdots \otimes x_1$ formally as the chain "$x_k \otimes | \otimes x_{k-1} \otimes \cdots \otimes x_1$, where all x_i (including x_k) carry their reduced degree, and $|$ carries degree 1. (so the pair "$x_k \otimes |$" carries degree $|x_k|$). This formal perspective is compatible with viewing the initial element x_k as belonging to the diagonal bimodule $\mathcal{C}_\Delta$, with its slightly sign-twisted bimodule multiplications (which are sign-twisted μ^k's in a manner compatible with this point of view; see [**G2**, §2.6]) rather than a morphism in the category $\mathcal{C}$. To illustrate this point of view in practice, note that the operator t, which involves permuting x_1 to the front of the chain and x_k to the right of the formal element $|$, must therefore come with a sign $||x_k|| + ||x_1||(\maltese_2^k + 1)$ as we wrote above. Similarly, all operations on a Hochschild chain, such as applying μ to a subsequence, or insertion of a unit, act "from the right" and hence come with Koszul reordering signs.

There are alternate, potentially simpler Koszul conventions for defining the Hochschild complex of the *shifted* diagonal bimodule, e.g., [**S3**, **S9**] but we have opted for the convention which is compatible with the existing literature on open-closed maps.

Unfortunately, Fukaya categories are typically not strictly unital (but rather, cohomologically unital), at least geometrically, so the complex $\mathrm{CH}_*(\mathcal{F}, \mathcal{F})$ does not have an as cleanly defined strict $\mathbf{k}[\Lambda]/\Lambda^2$ action. Instead, we consider the quasi-isomorphic *non-unital Hochschild complex*, which always carries a natural strict S^1 action. This complex seems to have been introduced by Tsygan [**T2**] and Loday-Quillen [**LQ**, §4] . We note that there are other methods of seeing the S^1 action on a Hochschild complex of $\mathcal{F}$, and at least one other method of constructing cyclic open-closed maps; see Remark 31.

As a graded vector space, the *non-unital Hochschild complex* consists of two copies of the ordinary Hochschild complex, the second copy shifted down in grading by 1:

$$\mathrm{CH}_*^{nu}(\mathcal{C}) := \mathrm{CH}_*(\mathcal{C}) \oplus \mathrm{CH}_*(\mathcal{C})[1] \tag{3.15}$$

With respect to the decomposition (3.15), we sometimes refer to elements as $\sigma := \check{\alpha} + \hat{\beta}$, with the notation $\hat{\alpha}$ or $\check{\beta}$ indicating that the element α and β belong to the left or right factor respectively. Similarly, we refer to the left and right factors as the *check factor* and the *hat factor* respectively.

Let b denote the usual Hochschild differential, and b' denote a version of the Hochschild differential omitting the "wrap-around terms" (this is simply often called the *bar differential*) as follows:

$$\begin{aligned} b'(x_d \otimes x_{d-1} \otimes \cdots \otimes x_1) = & \\ & \sum (-1)^{\maltese_1^s} x_d \otimes \cdots \cdots \otimes x_{s+j+1} \otimes \mu^j(x_{s+j} \otimes \cdots \otimes x_{s+1}) \otimes x_s \otimes \cdots \otimes x_1 \\ & + \sum (-1)^{\maltese_1^{d-j}} \mu^j(x_d, x_{d-1}, \ldots, x_{d-j+1}) \otimes x_{d-j} \otimes \cdots \otimes x_1. \end{aligned} \tag{3.16}$$

For an element $\hat{\beta} = x_d \otimes \cdots \otimes x_1$ in the hat (right) factor of the non-unital complex, define an element $d_{\wedge\vee}(\hat{\beta})$ in the check (left) factor of (3.15):

$$\begin{aligned} d_{\wedge\vee}(\hat{\beta}) &:= (-1)^{\maltese_2^d + ||x_1|| \cdot \maltese_2^d + 1} x_1 \otimes x_d \otimes \cdots \otimes x_2 + (-1)^{\maltese_1^{d-1}} x_d \otimes \cdots \otimes x_1. \\ &= (-1)^{\maltese_1^d + ||x_d||}(id - t)\bigg(x_d \otimes \cdots \otimes x_1\bigg). \end{aligned} \tag{3.17}$$

In this language, the differential on the non-unital Hochschild complex can be written as:

$$b^{nu} : (\check{\alpha}, \hat{\beta}) \mapsto (b(\check{\alpha}) + d_{\wedge\vee}(\hat{\beta}), b'(\hat{\beta})) \tag{3.18}$$

or equivalently can be expressed via the matrix

$$b^{nu} = \begin{pmatrix} b & d_{\wedge\vee} \\ 0 & b' \end{pmatrix}. \tag{3.19}$$

The inclusion $\mathrm{CH}_*(\mathcal{C},\mathcal{C}) \hookrightarrow \mathrm{CH}_*^{nu}(\mathcal{C},\mathcal{C})$ into the left factor is by construction a chain map. Since the quotient complex is the standard A_∞ bar complex with differential b', which is acyclic for homologically unital $\mathcal{C}$ (by a standard length filtration spectral sequence argument, see e.g., [**S5**, Lemma 2.12] or [**G2**, Prop. 2.2]), hence:

Lemma 3. *The inclusion $\iota : \mathrm{CH}_*(\mathcal{C}) \hookrightarrow \mathrm{CH}_*^{nu}(\mathcal{C})$ is a quasi-isomorphism (when $\mathcal{C}$ is cohomologically unital).* □

Remark 29. One way to obtain the non-unital Hochschild complex is as follows (compare [**L2**, §1.4.1]): first, augment the category $\mathcal{C}$ by adjoining strict unit morphisms e_X^+ to each $\hom_{\mathcal{C}}(X,X)$: in other words define a new A_∞ category $\mathcal{C}^+$ with ob $\mathcal{C}^+ =$ ob $\mathcal{C}$ and

$$\hom_{\mathcal{C}^+}(X,Y) = \begin{cases} \hom_{\mathcal{C}}(X,Y) & X \neq Y \\ \hom_{\mathcal{C}}(X,X) \oplus \mathbf{k}\langle e_X^+\rangle & X = Y \end{cases} \tag{3.20}$$

such that $\mathcal{C}$ is an A_∞ subcategory, and so that the e_X^+ elements act as strict units:

$$\begin{aligned} (-1)^{|y|}\mu^2(e_X^+, y) = \mu^2(y, e_X^+) = y \\ \mu^k(\cdots, e_X^+, \cdots) = 0,\ k > 2. \end{aligned} \tag{3.21}$$

(this completely determines all A_∞ operations in $\mathcal{C}^+$. Next, consider the *normalized Hochschild complex* $\mathrm{CH}_*^{red}(\mathcal{C}^+,\mathcal{C}^+)$ the quotient complex of $\mathrm{CH}_*(\mathcal{C})$ in which at most the starting term of a Hochschild chain is allowed to be an e_X^+. Then, take the further quotient by length 1 Hochschild chains of the form e_X^+ (so any chain beginning with e_X^+ must have length ≥ 2). The resulting complex, denoted $\widetilde{\mathrm{CH}}_*(\mathcal{C}^+)$, which is quasi-isomorphic to $\mathrm{CH}_*(\mathcal{C})$ when $\mathcal{C}$ has cohomological units, is isomorphic as a chain complex to $\mathrm{CH}_*^{nu}(\mathcal{C})$ via the following map $f : \widetilde{\mathrm{CH}}_*(\mathcal{C}^+) \to \mathrm{CH}_*^{nu}(\mathcal{C})$: If $\alpha = x_d \otimes \cdots \otimes x_1$ denotes a cyclically composible chain of morphisms in $\mathcal{C}$ with start object X, then

$$\begin{cases} f(e_X^+ \otimes \alpha) = \hat{\alpha} \\ f(\alpha) = \check{\alpha} \end{cases} \tag{3.22}$$

This map is a chain equivalence; in other words, the differential in CH^{nu} on a Hochschild chain $\hat{\alpha}$ agrees with the usual Hochschild differential applied to $e_X^+ \otimes \alpha$ using the rules for A_∞ multiplication with a strict unit (3.21).

In light of the previous remark, there is a natural operator B^{nu} on $\mathrm{CH}_*^{nu}(\mathcal{C})$ of degree -1, defined as

$$\begin{aligned} B^{nu}(x_k \otimes \cdots \otimes x_1, y_l \otimes \cdots \otimes y_1) &:= \sum_i (-1)^{\maltese_1^i \maltese_{i+1}^k + ||x_k|| + \maltese_1^k + 1}(0, x_i \otimes \cdots \otimes x_1 \otimes x_k \otimes \cdots \otimes x_{i+1})) \\ &= s^{nu} N, \end{aligned} \tag{3.23}$$

where $N(x_k \otimes \cdots x_1)$ is as in (3.12) and

$$s^{nu}(x_d \otimes \cdots \otimes x_1, y_t \otimes \cdots \otimes y_1) := (-1)^{\maltese_1^d + ||x_d|| + 1}(0, x_d \otimes \cdots \otimes x_1). \tag{3.24}$$

(in other words, $s^{nu}(x_i \otimes \cdots \otimes x_1 \otimes x_d \otimes \cdots \otimes x_{i+1}, y_t \otimes \cdots \otimes y_1) := (-1)^{\maltese_1^d + ||x_i|| + 1}(0, (x_i \otimes \cdots \otimes x_1 \otimes x_d \otimes \cdots \otimes x_{i+1})))$. Note that under the isomorphism f described in Remark 29, s^{nu} corresponds precisely to s defined in (3.13). In particular:

Lemma 4. *$(B^{nu})^2 = 0$ and $b^{nu}B^{nu} + B^{nu}b^{nu} = 0$. That is, $\mathrm{CH}_*^{nu}(\mathcal{C})$ is a strict S^1 complex, with the action of $\Lambda = [S^1]$ given by B^{nu}.* □

One can verify that when $\mathcal{C}$ is strictly unital, then there is a quasi-isomorphisms of S^1-complexes between $\mathrm{CH}_*(\mathcal{C})$ (with $b,\ B$) and $\mathrm{CH}_*^{nu}(\mathcal{C})$ (with b^{nu}, B^{nu}) (compare [**L2**, Thm. 2.1.8]). Let $b_{eq} = b^{nu} + uB^{nu}$ be the strict S^1-complex structure on the non-unital Hochschild complex $\mathrm{CH}_*^{nu}(\mathcal{C})$, u-linearly packaged as in §2.3. Using this, we can define cyclic homology groups:

Definition 5. *The* (positive) cyclic *chain complex, the* negative cyclic *chain complex, and the* periodic cyclic *chain complexes of* $\mathcal{C}$ *are the* homotopy orbit complex, homotopy fixed point complex, *and* Tate constructions *of the S^1-complex* $(\mathrm{CH}_*^{nu}(\mathcal{C}), b_{eq})$ *respectively. That is:*

$$\mathrm{CC}_*^+(\mathcal{C}) := (\mathrm{CH}_*^{nu}(\mathcal{C}))_{hS^1} = (\mathrm{CH}_*^{nu}(\mathcal{C}) \otimes_{\mathbf{k}} \mathbf{k}((u))/u\mathbf{k}[[u]], b_{eq}) \tag{3.25}$$

$$\mathrm{CC}_*^+(\mathcal{C}) := (\mathrm{CH}_*^{nu}(\mathcal{C}))^{hS^1} = (\mathrm{CH}_*^{nu}(\mathcal{C}) \widehat{\otimes}_{\mathbf{k}} \mathbf{k}[[u]], b_{eq}) \tag{3.26}$$

$$\mathrm{CC}_*^\infty(\mathcal{C}) := (\mathrm{CH}_*^{nu}(\mathcal{C}))^{Tate} = (\mathrm{CH}_*^{nu}(\mathcal{C}) \widehat{\otimes}_{\mathbf{k}} \mathbf{k}((u)), b_{eq}) \tag{3.27}$$

with grading induced by setting $|u| = +2$, and where (as in §2.3), $\widehat{\otimes}$ refers to the u-adically completed tensor product in the category of graded vector spaces. The cohomologies of these complexes, denoted $\mathrm{HC}_*^{+/-/\infty}(\mathcal{C})$*, are called the* (positive), negative, *and* periodic cyclic homologies *of* $\mathcal{C}$ *respectively.*

The $C_{-*}(S^1)$ module structure on $\mathrm{CH}_*^{nu}(\mathcal{C})$ is suitably functorial in the following sense: Let $\mathbf{F} : \mathcal{C} \to \mathcal{C}'$ be an A_∞ functor, There is an induced chain map on non-unital Hochschild complexes

$$\begin{aligned} \mathbf{F}_\sharp^{nu} : \mathrm{CH}_*^{nu}(\mathcal{C}) &\to \mathrm{CH}_*^{nu}(\mathcal{C}', \mathcal{C}') \\ (x, y) &\mapsto (\mathbf{F}_\sharp(x), \mathbf{F}'_\sharp(y)) \end{aligned} \tag{3.28}$$

where

$$\mathbf{F}'_\sharp(x_k \otimes \cdots \otimes x_0) := \sum_{i_1,\ldots,i_s} \mathbf{F}^{i_1}(x_k \cdots) \otimes \cdots \otimes \mathbf{F}^{i_s}(\cdots x_0) \tag{3.29}$$

$$\mathbf{F}_\sharp(x_k \otimes \cdots \otimes x_0) := \sum_{i_1,\ldots,i_s,j} \mathbf{F}^{j+1+i_1}(x_j, \ldots, x_0, x_k, \ldots, x_{k-i_1+1}) \otimes \mathbf{F}^{i_2}(\ldots) \otimes \cdots \otimes \mathbf{F}^{i_s}(x_{j+i_s} \ldots, x_{j+1}). \tag{3.30}$$

which is an isomorphism on homology if $\mathbf{F}$ is a quasi-isomorphism (indeed, even a Morita equivalence). This functoriality preserves S^1 structures:

Proposition 3. $\mathbf{F}_\sharp^{nu}$ *gives a strict morphism of strict S^1-complexes, meaning* $\mathbf{F}_\sharp^{nu} \circ b^{nu} = b^{nu} \circ \mathbf{F}_\sharp^{nu}$ *and* $\mathbf{F}_\sharp^{nu} \circ B^{nu} = B^{nu} \circ \mathbf{F}_\sharp^{nu}$*. In other words, the pre-morphism of A_∞ $\mathbf{k}[\Lambda]/\Lambda^2$ modules defined as*

$$\mathbf{F}_*^d(\underbrace{\Lambda, \ldots, \Lambda}_{d}, \sigma) := \begin{cases} \mathbf{F}_\sharp^{nu}(\sigma) & d = 0 \\ 0 & d \geq 1 \end{cases} \tag{3.31}$$

is closed, e.g., an A_∞ module homomorphism.

Sketch. It is well known that $\mathbf{F}_\sharp^{nu}$ is a chain map, so it suffices to verify that $\mathbf{F}_\sharp^{nu} \circ B^{nu} = B^{nu} \circ \mathbf{F}_\sharp^{nu}$, or in terms of (3.28)

$$\mathbf{F}'_\sharp \circ s^{nu} N = s^{nu} N \circ \mathbf{F}_\sharp. \tag{3.32}$$

We leave this an exercise, noting that applying either side to a Hochschild chain $x_k \otimes \cdots \otimes x_1$, the sums match identically. □

Remark 30. Continuing Remark 29, suppose we have constructed $\mathrm{CH}_*^{nu}(\mathcal{C})$ as $\widetilde{\mathrm{CH}}_*(\mathcal{C}) := \mathrm{CH}_*^{red}(\mathcal{C}^+)/ \oplus_X \mathbf{k}\langle e_X^+ \rangle$, the quotient of the reduced Hochschild complex of the augmented category

$\mathcal{C}^+$. Given any $\mathbf{F}$ as above, extend $\mathbf{F}$ to an augmented functor $\tilde{\mathbf{F}}$ by mandating that

$$\begin{aligned} \tilde{\mathbf{F}}^1(e_X^+) &= e_{\mathbf{F}X}^+ \\ \tilde{\mathbf{F}}^d(\ldots, e_X^+, \ldots) &= 0. \end{aligned} \tag{3.33}$$

Then $\mathbf{F}_\sharp^{nu}$ is just the morphism associated to $\tilde{\mathbf{F}}$ between reduced Hochschild complexes, and in particular Proposition 3 is a consequence of the corresponding statement that unital functors induce strict S^1-morphisms between reduced Hochschild complexes of unital categories.

Remark 31. There are options besides the non-unital Hochschild complex, for seeing the $C_{-*}(S^1)$ option on a Hochschild complex of the Fukaya category. For instance one could:

(1) perform a strictly unital replacement (via homological algebra as in [**S5**, §2] [**L1**, Thm. 3.2.1.1]), and work with the Hochschild complex of the replacement. However, this doesn't retain a relationship between the A_∞ operations and geometric structure, and hence is difficult to use with open-closed maps. Instead, one could:

(2) Geometrically construct a strictly unital structure on the Fukaya category via constructing *homotopy units* [**FOOO**], which roughly consist of a formal (geometrically defined) operation of multiplying by a strict unit, and higher homotopies between this operation and multiplying by the geometrically defined cohomological unit. The result is a strictly unital A_∞ category $\mathcal{F}^{hu}$ with $\hom_{\mathcal{F}^{hu}}(X,X) = \hom_{\mathcal{F}}(X,X) \oplus \mathbf{k}\langle e_X^+, f_X\rangle$, extending the A_∞ structure on $\mathcal{F}$, with e_X^+ a strict unit and $\mu^1(f_X) = e_X^+ - e_X$, e_X a chosen representative of a cohomological unit.

    Then, the usual Hochschild complex $\mathrm{CH}_*(\mathcal{F}^{hu}, \mathcal{F}^{hu})$ carries a strict S^1 action, and one can construct a cyclic open-closed map with source $\mathrm{CH}_*(\mathcal{F}^{hu}, \mathcal{F}^{hu})$, in a manner completely analogous to the construction of $\mathcal{F}^{hu}$. This option is equivalent to the one we have chosen (and has some benefits), but requires additional technical work.

    A geometric construction of homotopy units was introduced in the pioneering work of [**FOOO**, Ch. 7, §31]. See [**G1**] for an implementation in the wrapped, exact, multiple Lagrangians setting.

3.2. The Fukaya category. For the purposes of simplifying discussion, we focus on the technically simplest cases in which Fukaya categories can be defined, namely exact (Liouville) and monotone symplectic manifolds.

Definition 6. *An* admissible Lagrangian brane *consists of a properly embedded Lagrangian submanifold $L \subset M$, satisfying*

$$\begin{aligned} &\textit{exactness or monotonicity,} \\ &\textit{depending on the hypotheses imposed on } M \end{aligned} \tag{3.34}$$

and equipped with the following extra data (only required if one wants to work with $\operatorname{char} \mathbf{k} \neq 2$ *and* $\mathbb{Z}$ *gradings respectively):*

$$\textit{an orientation and Spin structure; and} \tag{3.35}$$

$$\textit{a grading in the sense of } [\mathbf{S1}]. \tag{3.36}$$

(these choices of extra data require L to be Spin and satisfy $2c_1(M, L) = 0$, where $c_1(M, L) \in H^2(M, L)$ is the relative first Chern class respectively).

Remark 32. The simplest version of the extra hypotheses (3.34) are as follows: if M is exact, one can require L to be exact, and come equipped with a fixed primitive $f_L : L \to \mathbb{R}$ vanishing away from a compact set.

Denote by $\operatorname{ob} \mathcal{F}$ a finite collection of admissible Lagrangian branes, which we simply refer to as Lagrangians. We choose a (potentially time-dependent) Hamiltonian $H_t : M \to \mathbb{R}$ satisfying the following non-degeneracy condition.

Assumption 1. *All time-1 chords of* X_{H_t} *between any pair of Lagrangians in* ob $\mathcal{F}$ *are non-degenerate.*

Remark 33. It is straightforward to adapt all of our constructions to larger collections of Lagrangians, by for instance, choosing a different Hamiltonian H_{L_0,L_1} for each pair of Lagrangians L_0, L_1, and by choosing Floer perturbation data depending on corresponding sequences of objects. We have opted for using a single H_t simply to keep the notation simpler.

For any pair of Lagrangians $L_0, L_1 \in \text{ob}\,\mathcal{F}$, the set of time 1 Hamiltonian flows of H, $\chi(L_0, L_1)$ can be again thought of as the critical points of an action functional on the *path space* from L_0 to L_1, $\mathcal{P}_{L_0,L_1}$ (this functional is most easily defined in the presence of primitives λ for ω and f_i for $\lambda|_{L_i}$). Using the extra data chosen for M in §4.1 and for each L_i above, elements of $\chi(L_0, L_1)$ can be graded by the *Maslov index*

$$\deg : \chi(L_0, L_1) \to \mathbb{Z}. \tag{3.37}$$

As a graded $\mathbf{k}$-module, the *morphism space* between L_0 and L_1, also known as the *Lagrangian Floer complex of* L_0 *and* L_1 *with respect to* H, has one (free) generator for each element of $\chi(L_0, L_1)$; concretely

$$\hom^i_{\mathcal{F}}(L_0, L_1) = CF^*(L_0, L_1, H_t, J_t) := \bigoplus_{x \in \chi(L_0,L_1), deg(x)=i} |o_x|_{\mathbf{k}}, \tag{3.38}$$

where the *orientation line* o_x is the real vector space associated to x by index theory (see [**S5**, §11h]) and $V_{\mathbf{k}}$ denotes the $\mathbf{k}$-normalization as in (4.5)).

The A_∞ structure maps arise as counts of parameterized families of solutions to Floer's equation with source a disc with d inputs and one output. For $d \geq 2$, we use the notation $\overline{\mathcal{R}}^d$ for the (Deligne-Mumford compactified) moduli space of discs with $d+1$ marked points modulo reparametrization, with one point z_0^- marked as negative and the remainder $z_1^+, \ldots, z_d^+$ (labeled counterclockwise from z_0^-) marked as positive. We orient the open locus of $\mathcal{R}^d$ as in [**S5**, §12g] and [**A**], by pulling back the standard orientation from a trivialization. $\overline{\mathcal{R}}^d$ can be given the structure of a manifold with corners, and its higher strata are trees of stable discs with a total of d exterior positive marked points and 1 exterior negative marked point.

Denote the positive and negative semi-infinite strips by

$$Z_+ := [0, \infty) \times [0, 1] \tag{3.39}$$

$$Z_- := (-\infty, 0] \times [0, 1] \tag{3.40}$$

One first equips the spaces $\overline{\mathcal{R}}^d$ for each d with a *consistent collection of strip-like ends* $\mathfrak{S}$: that is, for each component S of $\overline{\mathcal{R}}^d$, a collection of maps $\epsilon_k^\pm : Z_\pm \to S$ all with disjoint image in S, chosen so that positive/negative strips map to neighborhoods of positively/negatively-labeled boundary marked points respectively, smoothly varying with respect to the manifolds with corner structure and compatible with choices made on boundary and corner strata, which are products of lower dimensional copies of $\overline{\mathcal{R}}^k$'s.

In order to associate well-defined counts of transversely cut out moduli spaces to maps from such a parametrized family of domains, one can perturb Floer's equation by domain-dependent choices of data. The following description of the choices required is perhaps over-general, designed essentially to simultaneously work for the wrapped Fukaya category and compact Lagrangians:

Definition 7 (c.f. [**A**]). *A* Floer perturbation datum *adapted to a fixed pair* (H_t, J_t) *for a surface with boundary marked punctures equipped with fixed strip-like ends consists of the following choices:*

(1) For each strip-like end $\epsilon_k^\pm$*, a real number, called a* weight w_k*,*

(2) 1-form*: a one-form α_S restricting to $w_k dt$ on a strip-like end with weight w_k.*

$$(\epsilon_k^{\pm})^* \alpha_S = w_k dt$$

(3) *A* Hamiltonian $H_S : S \to \mathcal{H}(M)$*, restricting to* $\frac{H}{w_k^2}\psi^{w_k}$ *on a strip-like end with weight* w_k*:*

$$(\epsilon_k^{\pm})^* H_S = \frac{H}{w_k^2} \circ \psi^{w_k}$$

(4) *A* boundary-shifting map $\rho : \partial\bar{S}\backslash\{\textit{marked points}\} \to (0,\infty)$ *satisfying*

$$(\epsilon_k^{\pm})^* \rho = w_k.$$

(5) *An* S-dependent almost-complex structure J_S *satisfying*

$$(\epsilon_k^{\pm})^* J_S = (\psi^{w_k})^* J_t.$$

The operations associated to this family will come from (families of) solutions to a generalization of Floer's equation of maps from these domains into M. These equations, for a map $u : S \to M$, take the form

$$(du - X \otimes \alpha)^{0,1} = 0 \tag{3.41}$$

where X is the Hamiltonian vector field associated to a surface dependent Hamiltonian H_S, α is a one-form (e.g., dt for the usual Floer equation), and $(0,1)$ is with respect to some surface dependent almost complex structure. In order for a solution u to satisfy a strong form of the *maximum principle* (some form is required for solutions to not escape to ∞) as in [**AS**] [**A**] one sees that α must be sub-closed, i.e., $d\alpha \leq 0$, implying that it is not in general possible to have α restrict exactly to dt on inputs and outputs. The solution (as in the case of a linear Hamiltonians) is to allow α to be some multiple of wdt on each end. One still needs a way to identify the resulting complex with the usual Floer homology with respect to (H, J).

The solution comes via a *rescaling trick* first observed in [**FSS**] and systematically developed in [**A**]: first one notes that pullback of solutions to Floer's equation for (H, J_t) by the Liouville flow for time $\log(\rho)$ defines a canonical identification

$$CW^*(L_0, L_1; H, J_t) \simeq CW^*\left(\psi^\rho L_0, \psi^\rho L_1; \frac{H}{\rho} \circ \psi^\rho, (\psi^\rho)^* J_t\right). \tag{3.42}$$

The right hand object is equivalently the Floer complex for $(\psi^w L_0, \psi^w L_1)$ for a strip with one form wdt using Hamiltonian $\frac{H}{w^2} \circ \psi^w$ and $(\psi^w)^* J_t$. One can observe that

Lemma 5. *For any ρ, the function $\frac{H}{w^2} \circ \psi^w$ lies in $\mathcal{H}(M)$, i.e., is quadratic at ∞.*

Proof. The Liouville flow is given on the collar by

$$\psi^w(r, y) = (w \cdot r, y) \tag{3.43}$$

so $r^2 \circ \psi^w = w^2 r^2$. □

Recall that we have fixed a single Hamiltonian H and time-dependent almost complex structure J_t such that $CW^*(L_i, L_j, H, J_t)$ is defined for every $L_i, L_j \in \text{ob } \mathcal{W}$. We use the following symbols to refer to the (positive and negative) semi-infinite strips:

$$Z_+ := [0, \infty) \times [0, 1] \tag{3.44}$$

$$Z_- := (-\infty, 0] \times [0, 1] \tag{3.45}$$

Definition 8. *Floer datum A* Floer datum $\mathbf{F}_S$ *on a stable disc* $S \in \overline{\mathcal{R}}^d$ *consists of the following choices on each component:*

(1) *A collection of* strip-like ends $\mathfrak{S}$*; that is maps* $\epsilon_k^{\pm} : Z_+ \to S$ *all with disjoint image in* S*. These should be chosen so that positive strips map to neighborhoods of positively-labeled boundary marked points, and similarly for negative marked points.*

(2) *For each strip-like end* $\epsilon_k^{\pm}$*, a real number, called a* weight w_k,

(3) closed 1-form*: a one-form* α_S *satisfying* $d\alpha_S = 0$*, and* $(\alpha_S)|_{\partial S} = 0$*, restricting to* $w_k dt$ *on a strip-like end with weight* w_k*.*

$$(\epsilon_k^{\pm})^* \alpha_S = w_k dt$$

(4) *A* Hamiltonian $H_S : S \to \mathcal{H}(M)$*, restricting to* $\frac{H}{w_k^2}\psi^{w_k}$ *on a strip-like end with weight* w_k*:*

$$(\epsilon_k^{\pm})^* H_S = \frac{H}{w_k^2} \circ \psi^{w_k}$$

(5) *A* boundary-shifting map $\rho : \partial\bar{S} \backslash \{\textit{marked points}\} \to (0, \infty)$ *satisfying*

$$(\epsilon_k^{\pm})^* \rho = w_k.$$

(6) *An* almost-complex structure J_S *satisfying*

$$(\epsilon_k^{\pm})^* J_S = (\psi^{w_k})^* J_t.$$

Definition 9. *A pair of Floer data* $\mathbf{F}_S^1$*,* $\mathbf{F}_S^2$ *on* $S \in \overline{\mathcal{R}}^d$ *are said to be* conformally equivalent *if for some constant* K*,*

- *the strip-like ends coincide,*
- $\alpha_1 = K\alpha_2$*,*
- $\rho_1 = K\rho_2$
- $H_1 = \frac{H_2}{K} \circ \psi^K$*, and*
- $J_1 = (\psi^K)^* J_2$*.*

Definition 10. *A* consistent choice of Floer data for the A_∞ structure *is a(n inductive) choice of Floer data, for each* $d \geq 2$ *and for each representative* S *of* $\overline{\mathcal{R}}^d$*, smoothly varying in* S*, whose restriction to each boundary stratum is conformally equivalent to the product of Floer data coming from lower-dimensional spaces. With respect to the boundary gluing charts, the Floer data should agree to infinite order at boundary strata with Floer data obtained via gluing.*

Inductively, since there is a contractible space of choices consistent with lower levels, universal and consistent choices of Floer data for the A_∞ structure exist. Now let $L_0, \ldots, L_d$ be objects of $\mathcal{F}$, and consider a sequence of chords $\vec{x} = \{x_k \in \chi(L_{k-1}, L_k)\}$ as well as another chord $x_0 \in \chi(L_0, L_d)$. Given a fixed universal and consistent Floer data $\mathbf{D}_\mu$, write $\mathcal{R}^d(x_0; \vec{x})$ for the space of maps

$$u : S \to M$$

with source an arbitrary element $S \in \mathcal{R}^d$, satisfying moving boundary conditions and asymptotics

$$\begin{cases} u(z) \in \psi^{\rho_S(z)} L_k & \text{if } z \in \partial S \text{ lies between } z^k \text{ and } z^{k+1} \\ \lim_{s \to \pm\infty} u \circ \epsilon^k(s, \cdot) = x_k & \end{cases} \tag{3.46}$$

and differential equation

$$(du - X_S \otimes \alpha_S)^{0,1} = 0 \tag{3.47}$$

with respect to the complex structure J_S and Hamiltonian H_S.

The consistency of our Floer data with respect to the codimension one boundary of the abstract moduli spaces $\overline{\mathcal{R}}^d$ implies that the (Gromov-type) compactification $\overline{\mathcal{R}}^d(x_0; \vec{x})$ is obtained by adding the images of the natural inclusions

$$\overline{\mathcal{R}}^{d_1}(x_0; \vec{x}_1) \times \overline{\mathcal{R}}^{d_2}(y; \vec{x}_2) \to \overline{\mathcal{R}}^d(x_0; \vec{x}) \tag{3.48}$$

where y agrees with one of the elements of $\vec{x}_1$ and $\vec{x}$ is obtained by removing y from $\vec{x}_1$ and replacing it with the sequence $\vec{x}_2$. Here, we let d_1 range from 1 to d, with $d_2 = d - d_1 + 1$, with the stipulation that $d_1 =$ or $d_2 = 1$ is the semistable case:

$$\overline{\mathcal{R}}^1(x_0; x_1) := \overline{\mathcal{R}}(x_0; x_1) \tag{3.49}$$

Lemma 6. *For a generically chosen Floer data $\mathbf{D}_\mu$, the moduli space $\overline{\mathcal{R}}^d(x_0;\vec{x})$ is a smooth compact manifold of dimension*

$$\deg(x_0) + d - 2 - \sum_{1 \le k \le d} \deg(x_k),$$

and for fixed $\vec{x}$ is empty for all but finitely many x_0.

Proof. The *integrated maximum principle* of [**A**, §B] implies that elements of $\mathcal{R}(x_0;\vec{x})$ have image contained in a compact subset of M dependent on x_0 and $\vec{x}$ (this is strongly dependent on the form of H, J, and α chosen for our Floer data). The same result shows solutions do not exist for x_0 of sufficiently negative *action* compared to $\vec{x}$ (our conventions are such that the action is bounded above). The rest of the lemma follows from standard transversality methods and index calculations as in [**S5**, (9k), (11h), Prop. 11.13]. □

If $\deg(x_0) = 2 - d + \sum_1^d \deg(x_k)$, then the elements of $\overline{\mathcal{R}}^d(x_0;\vec{x})$ are rigid. By [**S5**, (11h), (12b),(12d)], given the fixed orientation of $\mathcal{R}^d$, any rigid element $u \in \overline{\mathcal{R}}^d(x_0;\vec{x})$ determines an isomorphism of orientation lines

$$\mathcal{R}^d_u : o_{x_d} \otimes \cdots \otimes o_{x_1} \longrightarrow o_{x_0}. \tag{3.50}$$

We define the dth A_∞ operation

$$\mu^d : CW^*(L_{d-1}, L_d) \otimes \cdots \otimes CW^*(L_0, L_1) \longrightarrow CW^*(L_0, L_d) \tag{3.51}$$

as a sum

$$\mu^d([x_d], \dots, [x_1]) := \sum_{\deg(x_0) = 2-d+\sum \deg(x_k)} \sum_{u \in \overline{\mathcal{R}}^d(x_0;\vec{x})} (-1)^{\bigstar_d} \mathcal{R}^d_u([x_d], \dots, [x_1]) \tag{3.52}$$

where the sign is given by

$$\bigstar_d = \sum_{i=1}^d i \cdot \deg(x_i) \tag{3.53}$$

(note that this sum is finite by Corollary 6). An analysis of the codimension 1 boundary of 1-dimensional moduli spaces along with their induced orientations establishes

Lemma 7 ([**S5**, Prop. 12.3]). *The maps μ^d satisfy the A_∞ relations.* □

4. Circle action on the closed sector

4.1. Floer cohomology and symplectic cohomology. Let (M^{2n}, ω) be a symplectic manifold satisfying some technical hypotheses (such as Liouville, monotone, etc.—whichever hypotheses satisfy the Assumptions stated below). Given a (potentially time-dependent) Hamiltonian $H : M \to \mathbb{R}$, *Hamiltonian Floer cohomology* when it is defined is formally the Morse cohomology of the H-perturbed action functional on the free loop space of M: $\mathcal{A}_H : \mathcal{L}M \to \mathbb{R}$. If ω is exact and comes with a fixed primitive λ, this functional can be written as:

$$x \mapsto -\int_x \lambda + \int_0^1 H_t(x(t))dt$$

In general, $\mathcal{A}_H$ may be multi-valued, but $d\mathcal{A}_H$ is always well-defined, leading at least to a Morse-Novikov type theory. We write $\mathcal{O}$ for the set of *critical points* of $\mathcal{A}_{H_t}$ (when H_t is implicit), which are precisely the time-1 orbits of the associated Hamiltonian vector field X_H.

Assumption 2. *The elements of $\mathcal{O}$ are non-degenerate.*

Optionally, fix the data of a trivialization of a quadratic complex volume form on M, which allows us to define an absolute $\mathbb{Z}$ grading on orbits by $\deg(y) := n - CZ(y)$, where CZ is the Conley-Zehnder index of y.

Fix (potentially S^1-dependent) almost complex structure J_t. In the formal picture, this induces a metric on $\mathcal{L}M$. A *Floer trajectory* is formally a gradient flowline of $\mathcal{A}_{H_t}$ using the metric induced by J_t; concretely it is a map $u : (-\infty, \infty) \times S^1 \to M$, satisfying the following gradient flow equation for $\mathcal{A}_{H_t}$, also known as as *Floer's equation*:

$$(du - X_H \otimes dt)^{0,1} = 0 \tag{4.1}$$

and converging exponentially near $\pm\infty$ to a pair of specified orbits $y^{\pm} \in \mathcal{O}$. In standard coordinates s, t on the cylinder this reads as

$$\partial_s u = -J_t(\partial_t u - X). \tag{4.2}$$

The following crucial hypothesis is trivial if M is compact, but otherwise will not necessarily hold for arbitrary $(H_t,\ J_t)$:

Assumption 3 (A priori compactness estimate). *For fixed asymptotics y^+, y^-, there is a compact set $K \subset M$ depending only on $x^{\pm}$, H, and J, such that any Floer trajectory u between $x^{\pm}$ has image contained in K. Moreoever, from a given y^+, there are no Floer trajectories to all but finitely many y^-.*

The space of non-constant Floer trajectories between a fixed y^+ and y^- modulo the free $\mathbb{R}$ action given by translation in the s direction is denoted $\mathcal{M}(y^-; y^+)$. As in Morse theory, one should compactify this space by allowing *broken trajectories*:

$$\overline{\mathcal{M}}(y^-; y^+) = \coprod \mathcal{M}(y^-; y^1) \times \mathcal{M}(y^1; y^2) \times \cdots \times \mathcal{M}(y^k; y^+). \tag{4.3}$$

The next hypothesis stipulates that aside from the above Morse-type breakings of trajectories, there are no "bad" breakings (such as sphere bubbles), and moreoever that the actual and formal dimensions (index-theoretic) dimensions of these moduli spaces agree:

Assumption 4 (Transversality and compactness). *For generic choices of (time-dependent) J_t, $\overline{\mathcal{M}}(y^-; y^+)$ is a compact manifold (with boundary) of dimension $\deg(y^-) - \deg(y^+) - 1$, at least whenever this $\deg(y^-) - \deg(y^+) \leq 2$.*

Putting this all together, the *Floer co-chain complex* for (H_t, J_t) over $\mathbf{k}$ has generators corresponding to orbits of H_t:

$$CF^i(M; H_t, J_t) = \bigoplus_{y \in \mathcal{O}, deg(y) = i} |o_y|_{\mathbf{k}}, \tag{4.4}$$

where the *orientation line* o_y is a real vector space associated to every orbit in $\mathcal{O}$ via index theory (see e.g., [**A**, §C.6]) and for any one dimensional real vector space V and any ring $\mathbf{k}$, the $\mathbf{k}$-*normalization*

$$|V|_{\mathbf{k}} \tag{4.5}$$

is the $\mathbf{k}$-module generated by the two possible orientations on V, with the relationship that their sum vanishes (if one does not want to worry about signs, note that $|V|_{\mathbb{Z}/2} \cong \mathbb{Z}/2$ canonically). The differential $d : CF^*(M; H_t, J_t) \longrightarrow CF^*(M; H_t, J_t)$ counts rigid elements of the compactified moduli spaces. To fix sign issues, we recall that for regular elements $u \in \mathcal{M}(y_0; y_1)$ with $\deg(y_0) = \deg(y_1) + 1$ (so u is rigid), there are natural isomorphisms between orientation lines induced by index theory (see e.g., [**S5**, (11h), (12b),(12d)], [**A**, Lemma C.4])

$$\mu_u : o_{y_1} \longrightarrow o_{y_0}. \tag{4.6}$$

Then, one defines the differential as

$$d([y_1]) = \sum_{y_0; \deg(y_0) = \deg(y_1) + 1} \ \sum_{u \in \overline{\mathcal{M}}(y_0; y_1)} (-1)^{\deg(y_1)} \mu_u([y_1]). \tag{4.7}$$

Under the assumptions made above, d is well-defined and $d^2 = 0$, and one calls the resulting group $HF^*(H_t, J_t)$.

Remark 34. Our (cohomological) grading convention for Floer cohomology follows [**S4**, **R**, **A**, **G1**].

4.1.1. *Symplectic cohomology. Symplectic cohomology* [**CFH**, **CFHW**, **FH**, **V**], is Hamiltonian Floer cohomology for a particular class of Hamiltonians on non-compact convex symplectic manifolds. There are several methods for defining this group. We define it here by making the following specific choices of target, Hamiltonian, and almost complex structure:

- M is a *Liouville manifold equipped with a conical end*, meaning that it comes equipped with a fixed generic primitive λ of ω, a compact subdomain with boundary $\bar{M} \subset M$, and a *conical structure* outside $\bar{M}$:
$$M \backslash \bar{M} = ([1, +\infty)_r \times \partial \bar{M}, r\lambda_{\partial \bar{M}}), \tag{4.8}$$
such that the associated Liouville vector field Z is outward pointing along $\partial \bar{M}$, and has the form $r\partial_r$ on (4.8). The conical structure (4.8) serves primarily as a technical device (it is known that the resulting invariants are independent of the specific such choice made).
- The Hamiltonian term H_t is a sum $H + F_t$ of an *autonomous Hamiltonian* $H : M \to \mathbb{R}$ which is *quadratic at* ∞, namely
$$H|_{M \backslash \bar{M}}(r, y) = r^2, \tag{4.9}$$
and a time-dependent perturbation F_t such that on the collar (4.8) of M,
$$\text{for any } r_0 \gg 0, \text{ there exists an } R > r_0 \text{ such that } F(t, r, y) \text{ vanishes in a neighborhood of } R. \tag{4.10}$$
(for instance, F_t could be supported near non-trivial orbits of H, where it is modeled on a Morse function on the circle). We denote by $\mathcal{H}(M)$ the class of Hamiltonians satisfying (4.9).
- The almost complex structure should begin to the class $\mathcal{J}(M)$ of complex structures which are *rescaled contact type* on the conical end, meaning that for some $c > 0$,
$$\lambda \circ J = f(r)dr \tag{4.11}$$
where f is any function with $f(r) > 0$ and $f'(r) \geq 0$.

The following Proposition is well known, and in the formulation described here, appears in e.g., [**R**, **A**]:

Proposition 4. *Assumptions 2, 3, and 4 hold for this M equipped with sufficiently generic choices of Liouville 1-form λ, H_t, and J_t.*

If M, H_t, and J_t are specifically as above, we refer to the Floer co-chain complex $CH^*(M, H_t, J_t)$ and J as above as the *symplectic co-chain complex* $SC^*(M)$ and the resulting cohomology group as *symplectic cohomology* $SH^*(M)$.

4.1.2. *Relative cohomology.* If instead we took

- M to be either a *Liouville manifold* or monotone symplectic manifold
- and H_t to be a non-degenerate Hamiltonian satisfying the following if M is non-compact (and arbitrary otherwise):
$$H_t|_{M \backslash \bar{M}}(r, y) = -\lambda r \tag{4.12}$$
where $\lambda \ll 1$ is a sufficiently small number (smaller than the length of any Reeb orbit on $\partial \bar{M}$); and
- Restrict the allowable J_t to be of *rescaled contact type* as before (which is an empty condition if M is compact).

The following Proposition is well known:

Proposition 5. *Assumptions 1, 2, and 3 hold, and (for generic J_t when $HF^*(H_t, J_t)$ is defined) there is an isomorphism $HF^*(H_t, J_t) \cong H^*(\bar{M}, \partial\bar{M})$.* $\square$

The isomorphism can be realized in one of two ways:

- choose H to be a C^2 small Morse function, in which case a well-known argument of Floer [**F1**] equates $HF^*(H_t, J_t)$ with the Morse complex of H,
- There is a geometric *PSS morphism* [**PSS**] $PSS : H_{2n-*}(M) \to HF^*(H_t, J_t)$, whose description we omit here.

4.2. The cohomological BV operator. The first order BV operator is a Floer analogue of a natural operator that exists on the Morse cohomology of any manifold with a smooth S^1 action. Like the case of ordinary Morse theory, this operator exists even when the Hamiltonian and complex structure (c.f. Morse function and metric) are not S^1-equivariant.

For $p \in S^1$, consider the following collection of cylindrical ends on $\mathbb{R} \times S^1$:

$$\begin{aligned} \epsilon_p^+ &: (s,t) \mapsto (s+1, t+p), \quad s \geq 0 \\ \epsilon_p^- &: (s,t) \mapsto (s-1, t), \quad s \leq 0 \end{aligned} \tag{4.13}$$

Pick $K : S^1 \times (\mathbb{R} \times S^1) \times M \to \mathbb{R}$ dependent on p, satisfying

$$(\epsilon_p^{\pm})^* K(p, s, \cdot, \cdot) = H(t, m) \tag{4.14}$$

meaning that

$$K_p(s,t,m) = \begin{cases} H(t+p, m) & s \geq 1 \\ H(t,m) & s \leq -1, \end{cases} \tag{4.15}$$

so in the range $-1 \leq s \leq 1$, $K_p(s,t,m)$ interpolates between $H_{t+p}(m)$ and $H_t(m)$ (and outside of this interval is independent of s).

Similarly, pick a family of almost complex structures $J : S^1 \times (\mathbb{R} \times S^1) \times M \to \mathbb{R}$ satisfying

$$(\epsilon_p^{\pm})^* J(p, s, t, m) = J(t, m) \tag{4.16}$$

Now, $x^+, x^- \in \mathcal{O}$, define

$$\mathcal{M}_{-1}(x^+, x^-) \tag{4.17}$$

to be the following *parametrized moduli space* of Floer cylinders

$$\{p \in S^1, u : S \to M \mid \begin{cases} \lim_{s \to \pm\infty} (\epsilon_p^{\pm})^* u(s, \cdot) = x^{\pm} \\ (du - X_K \otimes dt)^{0,1} = 0. \end{cases} \} \tag{4.18}$$

There is a natural bordification by adding broken Floer cylinders to either end

$$\overline{\mathcal{M}}_{-1}(x^+, x^-) = \coprod \mathcal{M}(x^+; a_0) \times \cdots \times \mathcal{M}(a_{k-1}; a_k) \times \mathcal{M}_{-1}(a_k, b_1) \times \mathcal{M}(b_1; b_2) \times \cdots \times \mathcal{M}(b_l; x^-) \tag{4.19}$$

For non-compact M, the following hypothesis is important to verify:

Assumption 5 (A priori compactness estimate). *The elements of $\mathcal{M}_{-1}(x^+, x^-)$ satisfy an* a priori compactness estimate*, meaning for fixed asymptotics y^+, y^-, there is a compact set $K \subset M$ depending only on $x^{\pm}$, H, and J, such that any Floer trajectory u between $x^{\pm}$ has image contained in K. Also, from a given y^+, there are no Floer trajectories to all but finitely many y^-.*

Remark 35. In the case of symplectic cohomology Hamiltonians described in §4.1.1, Assumption 5 can be ensured by choosing K carefully as follows. Given that $H_t(M) = H + F_t$ is a sum of an autonomous term and a time-dependent term that is zero at infinitely many levels tending towards infinity, we can ensure that

$$\text{at infinity many levels tending towards infinity, } K_p(s,t,m) \text{ is equal to } r^2, \tag{4.20}$$

and in particular is autonomous. The proof of Assumption 5 follows from [**A**, §B].

Assumption 6 (Transversality and compactness). *For generic choices of the above data,* $\overline{\mathcal{M}}_{-1}$ *is a smooth compact manifold with boundary of dimension* $\deg(x_+) - \deg(x^-) + 1$ *(at least whenever this numerical quantity is* ≤ 2*), with codimension 1 boundary covered by the closure of inclusions of strata* (4.19) *consisting of a single breaking.*

In the usual fashion, counts associated to the compactified moduli space with the right sign (which we explain more carefully in the next section) give an operation δ_1, satisfying

$$d\delta_1 + \delta_1 d = 0.$$

It would be desirable for δ_1 square to zero on the chain level, which would give $(SC^*(M), d, \delta_1)$ the structure of a *strict* S^1*-complex*, or *mixed complex*. However, the S^1 dependence of our Hamiltonian and almost complex structure prevent this, in a manner we now explain.

Typically one attempts to prove an geometric/Floer-theoretic operation (such as δ_1^2) is zero by exhibiting that the relevant moduli problem has no zero-dimensional solutions (due to, say, extra symmetries in the equation), or otherwise arises as the boundary of a 1-dimensional moduli space. To that end, we first indicate a moduli space parametrized by $S^1 \times S^1$ which looks like two of the previous parametrized spaces naively superimposed, leading us to call the associated operation we call δ_2^{naive}. The extra symmetry involved in this definition will allow us to easily conclude

Lemma 8. δ_2^{naive} *is the zero operation.*

For $(p_1, p_2) \in S^1 \times S^1$, consider the following collection of cylindrical ends:

$$\begin{aligned} &\epsilon^+_{(p_1,p_2)} : (s,t) \mapsto (s+1, t+p_1+p_2), \quad s \geq 0 \\ &\epsilon^-_{(p_1,p_2)} : (s,t) \mapsto (s-1, t), \quad s \leq 0 \end{aligned} \tag{4.21}$$

Pick $K : (S^1 \times S^1) \times (\mathbb{R} \times S^1) \times M \to \mathbb{R}$ dependent on (p_1, p_2), satisfying

$$\epsilon^{\pm}_{(p_1,p_2)} K(p_1, p_2, s, \cdot, \cdot) = H(t, m) \tag{4.22}$$

meaning that

$$K_{(p_1,p_2)}(s,t,m) = \begin{cases} H(t+p_1+p_2, m) & s \geq 1 \\ H(t,m) & s \leq -1, \end{cases} \tag{4.23}$$

so in the range $-1 \leq s \leq 1$, $K_{p_1+p_2}(s,t,m)$ interpolates between $H_{t+p_1+p_2}(m)$ and $H_t(m)$.

Similarly, pick a family of almost complex structures $J : S^1 \times S^1 \times (\mathbb{R} \times S^1) \times M \to \mathbb{R}$

$$\epsilon^{\pm}_{(p_1,p_2)} J(p_1, p_2, s, t, m) = J(t, m), \tag{4.24}$$

such that

$$J \text{ only depends on the sum } p_1 + p_2. \tag{4.25}$$

Now, $x^+, x^- \in \mathcal{O}$, define

$$\mathcal{M}^{naive}_{-2}(x, y) \tag{4.26}$$

to be the *parametrized moduli space* of Floer cylinders

$$\{(p_1,p_2) \in S^1 \times S^1, u : S \to M \mid \begin{cases} \lim_{s \to \pm\infty} (\epsilon^{\pm}_{(p_1,p_2)})^* u(s, \cdot) = x^{\pm} \\ (du - X_K \otimes dt)^{0,1} = 0. \end{cases} \} \tag{4.27}$$

For generic choices of K and J, this moduli space, suitably compactified by adding broken trajectories, will be a manifold of dimension $\deg(x) - \deg(y) + 2$ (the details are similar to the previous section, and will be omitted). Counts of rigid elements in this moduli space will thus, in the usual fashion give a map of degree -2, which we call δ_2^{naive}.

Proof of Lemma 8. Let (p_1, p_2, u) be an element of $\mathcal{M}_{-2}^{naive}(x, y)$. Then, $(p_1 - r, p_2 + r, u)$ is an element too, for any $r \in S^1$, as the equation satisfied by the map u only depends on the sum $p_1 + p_2$. We conclude that elements of $\mathcal{M}_{-2}^{naive}(x, y)$ are never rigid, and thus that the resulting operations δ_2 is zero. □

We would like δ_2^{naive} to be genuinely equal to δ_1^2, which would imply that $\delta_1^2 = 0$ However, this is only true on the homology level; the lack of S^1 invariance of our Hamiltonian and almost complex structure, and the corresponding family of choices of homotopy between $\theta^* H_t$ and H_t, over varying θ , breaks symmetry and ensures that $\delta_1^2 \neq \delta_2^{naive}$ as geometric chain maps. However, there is a geometric chain homotopy, δ_2 between δ_1^2 and δ_2^{naive} along with a hierarchy of higher homotopies δ_k forming the S^1-complex structure on $CF^*(M)$, which we define in the next section.

4.3. The chain-level circle action. We turn to a "coordinate-free" definition of the relevant parametrized moduli spaces, which will help us incorporate the construction into open-closed maps.

Definition 11. *A* r-point angle-decorated cylinder *consists of a semi-infinite or infinite cylinder* $C \subseteq (-\infty, \infty) \times S^1$*, along with a collection of auxiliary points* $p_1, \ldots, p_r \in C$*, satisfying*

$$(p_1)_s \leq \cdots \leq (p_r)_s, \tag{4.28}$$

where $(a)_s$ *denotes the* $s \in (-\infty, \infty)$ *coordinate. The* heights *associated to this data are the* s *coordinates*

$$h_i = (p_i)_s, \ i = 1, \ldots, r \tag{4.29}$$

and the angles *associated to* C *are the* S^1 *coordinates*

$$\theta_i := (p_1)_t, \ i \in 1, \ldots, r. \tag{4.30}$$

The *cumulative rotation* of an r-point angle-decorated cylinder is the first angle:

$$\eta := \eta(C, p_1, \ldots, p_r) = \theta_1. \tag{4.31}$$

The *ith incremental rotation* of an r-point angle-decorated cylinder is the difference between the ith and $i-1$st angles:

$$\kappa_i^{inc} := \theta_i - \theta_{i+1} \text{ (where } \theta_{r+1} = 0). \tag{4.32}$$

Definition 12. *The* moduli space of r-point angle-decorated cylinders

$$\mathcal{M}_r \tag{4.33}$$

is the space of r*-point angle-decorated infinite cylinders, modulo translation.*

Remark 36 (Orientation for $\mathcal{M}_r$). Note that C_r, the space of all r-point angle-decorated infinite cylinders (not modulo translation) has a canonical complex orientation. Thus, to orient the quotient space $\mathcal{M}_r := C_r/\mathbb{R}$ it is sufficient to give a choice of trivialization of the action of $\mathbb{R}$ on C_r. We choose ∂_s to be the vector field inducing said trivialization.

For an element of this moduli space, the angles and relative heights of the auxiliary points continue to be well-defined, so there is a non-canonical isomorphism

$$\mathcal{M}_r \simeq (S^1)^r \times [0, \infty)^{r-1} \tag{4.34}$$

The moduli space $\mathcal{M}_r$ thus possesses the structure of an open manifold with corners, with boundary and corner strata given by the various loci where heights of the auxiliary points p_i are coincident (we allow the points p_i themselves to coincide; one alternative is to first Deligne-Mumford compactify, and then collapse all sphere bubbles containing multiple $p_i's$. That the result still forms a smooth manifold with corners is a standard local calculation near any such stratum). Given an arbitrary representative C of $\mathcal{M}_r$ with associated heights $h_1, \ldots, h_r$, we can always find a translation $\tilde{C}$ satisfying $\tilde{h}_r = -\tilde{h}_1$; we call this the *standard representative* associated to C.

Given a representative C of this moduli space, and a fixed constant δ, we fix a positive cylindrical end around $+\infty$

$$\begin{aligned}\epsilon^+ : [0, \infty) \times S^1 &\to C \\ (s,t) &\mapsto (s + h_r + \delta, t)\end{aligned} \tag{4.35}$$

and a negative cylindrical end around $-\infty$ (note the angular rotation in t!):

$$\begin{aligned}\epsilon^- : (-\infty, 0] \times S^1 &\to C \\ (s,t) &\mapsto (s - (h_1 - \delta)), t + \theta_1).\end{aligned}$$

These ends are disjoint from the p_i and vary smoothly with C; via thinking of C as a sphere with two points with asymptotic markers removed, these cylindrical ends correspond to the positive asymptotic marker having angle 0 and the negative asymptotic marker having angle $\theta_1 = \kappa_1^{inc} + \kappa_2^{inc} + \cdots + \kappa_r^{inc}$.

There is a compactification of $\mathcal{M}_r$ consisting of *broken r-point angle-decorated cylinders*

$$\overline{\mathcal{M}}_r = \coprod_s \coprod_{j_1, \ldots, j_s; j_i > 0, \sum j_i = r} \mathcal{M}_{j_1} \times \cdots \times \mathcal{M}_{j_s}. \tag{4.36}$$

The stratum consisting of s-fold broken configurations lies in the codimension s boundary, with the manifolds-with-corners structure explicitly defined by local gluing maps using the ends (4.35) and (4.36). Note that the gluing maps, which rotate the bottom cylinder in order to match an end (4.35) with (4.36), induce cylindrical ends on the glued cylinders which agree with the choices of ends made in (4.35)-(4.36).

The compactification $\overline{\mathcal{M}}_r$ thus has codimension-1 boundary covered by the images of the natural inclusion maps

$$\overline{\mathcal{M}}_{r-k} \times \overline{\mathcal{M}}_k \longrightarrow \partial \mathcal{M}_r, \; 0 < k < r \tag{4.37}$$

$$\overline{\mathcal{M}}_r^{i,i+1} \longrightarrow \partial \mathcal{M}_r, \; 1 \leq i < r, \tag{4.38}$$

where $\overline{\mathcal{M}}_r^{i,i+1}$ denotes the compactification of the locus where ith and $i+1$st heights are coincident

$$\mathcal{M}_r^{i,i+1} := \{C \in \mathcal{M}_r \mid h_i = h_{i+1}\}. \tag{4.39}$$

With regards to the above stratum, for $r > 1$ there is a projection map which will be relevant, a version of the forgetful map which remembers only the first of the angles with coincident heights:

$$\begin{aligned}\pi_i : \mathcal{M}_r^{i,i+1} &\longrightarrow \mathcal{M}_{r-1} \\ (h_1, \ldots, h_i, h_{i+1} = h_i, h_{i+2}, \ldots, h_r) &\longmapsto (h_1, \ldots, h_i, h_{i+2}, \ldots, h_r) \\ (\theta_1, \ldots, \theta_i, \theta_{i+1}, \ldots, \theta_r) &\longmapsto (\theta_1, \ldots, \theta_{i-1}, \theta_i, \widehat{\theta_{i+1}}, \theta_{i+2}, \ldots, \theta_r).\end{aligned} \tag{4.40}$$

π_i is compatible with the choice of positive and negative ends (4.35)-(4.36) and hence π_i extends to compactifications

$$\pi_i : \overline{\mathcal{M}}_r^{i,i+1} \to \overline{\mathcal{M}}_{r-1}. \tag{4.41}$$

Definition 13. *A* Floer datum for an r-point angle-rotated cylinder $\tilde{C} := (C, p_1, \ldots, p_r)$ *consists of the following choices:*

- *The positive and negative strip-like ends on $\epsilon^\pm : C^\pm \to C$ chosen in* (4.35)-(4.36).
- *The one-form on C given by $\alpha = dt$.*
- *A surface-dependent Hamiltonian $H_{\tilde{C}} : C \to \mathcal{H}(M)$ compatible with the positive and negative strip-like ends, meaning that*

$$(\epsilon^\pm)^* H_C = H_t, \tag{4.42}$$

where H_t was the previously chosen Hamiltonian.

- *A surface dependent complex structure $J_{\tilde{C}} : C \to \mathcal{J}_1(M)$ also compatible with $\epsilon^{\pm}$, meaning that*

$$(\epsilon^{\pm})^* J_{\tilde{C}} = J_t \tag{4.43}$$

  for our previously fixed choice J_t.

Definition 14. *A* universal and consistent choice of Floer data for the S^1 *action is an inductive choice of Floer data, for each k and each representative $S = (C, p_1, \ldots, p_k)$ of $\overline{\mathcal{M}}_k$, satisfying the following conditions at boundary strata:*

- *At the boundary strata* (4.37)*, the chosen data equals the product of Floer data already chosen on lower-dimensional spaces. Moreover, the choices made smoothly vary with respect to the gluing charts.*
- *At a boundary stratum of the form* (4.38)*, the Floer datum for S is conformally equivalent to one pulled back from $\overline{\mathcal{M}}_{k-1}$ via the forgetful map π_i, defined in* (4.41)*.*

Inductively, since the space of choices at each level is non-empty and contractible, universal and consistent choices of Floer data exist. From the gluing map, a representative S sufficiently near the boundary strata (4.37) inherits cylindrical regions, also known as *thin parts*, which are the surviving images of the cylindrical ends of lower-dimensional strata. Together with the cylindrical ends of S, this determines a collection of cylindrical regions.

Picking a universal and consistent choice of Floer data for the weak S^1 action, for $(x^+, x^-) \in \mathcal{O}$, we define for each $k \geq 1$,

$$\mathcal{M}_k(x^+; x^-) \tag{4.44}$$

to be the parametrized space of maps

$$\{S = (C, p_1, \ldots, p_r) \in \mathcal{M}_k,\ u : C \to M | \begin{cases} \lim_{s \to \pm\infty} (\epsilon^{\pm})^* u(s, \cdot) & = x^{\pm} \\ (du - X_{H_S} \otimes dt)^{(0,1)_S} & = 0. \end{cases} \tag{4.45}$$

meaning that u solves Floer's equation with respect to the Hamiltonian H_S and complex structure J_S chosen for the given element S. Standard methods imply that boundary of the Gromov bordification $\overline{\mathcal{M}}_k(x^+; x^-)$ is covered by the images of the natural inclusions

$$\overline{\mathcal{M}}_r(y; x^-) \times \overline{\mathcal{M}}_{k-r}(x^+; y) \to \partial \overline{\mathcal{M}}_k(x^+; x^-) \tag{4.46}$$

$$\overline{\mathcal{M}}_k^{i,i+1}(x^+; x^-) \to \partial \overline{\mathcal{M}}_k(x^+; x^-), \tag{4.47}$$

along with the usual semi-stable strip breaking boundaries

$$\begin{aligned} \overline{\mathcal{M}}_k(y; x^-) \times \overline{\mathcal{M}}(x^+; y) \to \partial \overline{\mathcal{M}}_k(x^+; x^-) \\ \overline{\mathcal{M}}(y; x^-) \times \overline{\mathcal{M}}_k(x^+; y) \to \partial \overline{\mathcal{M}}_k(x^+; x^-) \end{aligned} \tag{4.48}$$

Standard methods establish that

Lemma 9. *For generic choices of Floer data for the weak S^1 action, the moduli spaces $\overline{\mathcal{M}}_k(x^+; x^-)$ are smooth compact manifolds of dimension*

$$\deg(x^+) - \deg(x^-) + (2k - 1). \tag{4.49}$$

□

As usual, signed counts of rigid elements of this moduli space for varying x^+ and x^- (using induced maps on orientation lines, twisted as in the differential by $(-1)^{\deg(x_+)}$—see (4.7)) give the matrix coefficients for the overall map

$$\delta_k : CF^*(M) \to CF^{*-2k+1}(M). \tag{4.50}$$

Lemma 10. *For each k,*

$$\sum_{i=0}^{k} \delta_i \delta_{k-i} = 0. \tag{4.51}$$

Proof. The counts of rigid elements associated to the boundary of 1-dimensional components of $\partial\overline{\mathcal{M}}_k(x^+; x^-)$, along with a description of this codimension 1 boundary (4.46)-(4.48) immediately implies that

$$(\sum_{i=1}^{k} \delta_i \delta_{k-i}) + (\sum_i \delta_k^{i,i+1}) + (d\delta_k + \delta_k d) = 0, \tag{4.52}$$

where $\delta_k^{i,i+1}$ for each i is the operation associated to the moduli space of maps (4.47). (Observe that $\delta_2^{1,2}$ is precisely the operation δ_2^{naive} from §4.2). Note that the consistency condition for Floer data implies that the Floer data chosen for elements $S \in \mathcal{M}_k^{i,i+1}$ only depends on $\pi_i(S)$, where the forgetful map $\pi_i : \mathcal{M}_k^{i,i+1} \to \mathcal{M}_{k-1}$ has one-dimensional fibers. Hence given an element $(S, u) \in \overline{\mathcal{M}}_k^{i,i+1}(x^+; x^-)$, and a point $S' \in \mathcal{M}_k^{i,i+1}$ in the same fiber of π_i as S, (S', u) is another element of $\overline{\mathcal{M}}_k^{i,i+1}(x^+; x^-)$. In other words, elements of $\overline{\mathcal{M}}_k^{i,i+1}(x^+; x^-)$ are never rigid, so the associated operation $\delta_k^{i,i+1}$ is zero. □

The discussion in §2.1 implies then that

Corollary 4. *The pair $(CF^*(M; H_t, J_t), \{\delta_k\}_{k\geq 0})$ as defined above form an an S^1-complex.* □

In a standard way (see e.g., [**S5**, §10] [**Z**], by using continuation maps parametrized by various $(S^1)^k \times (0, 1]^k$ (or equivalently, by spaces of angle-decorated cylinders, not modulo $\mathbb{R}$), to prove that

Proposition 6. *Any continuation map $f : CF^*(M, H_1) \to CF^*(M, H_2)$ enhances to a homomorphism $\mathbf{F}$ of S^1-complexes. In particular, if $[f]$ is an isomorphism, then there is a corresponding quasi-isomorphism of S^1-complexes.*

In particular, the S^1-complex defined on the symplectic co-chain complex $SC^*(M)$ or the Hamiltonian Floer complex (with small negative slope if M is non-compact) is an invariant of M, up to quasi-isomorphism.

4.3.1. *Relation to earlier definitions in the literature.* There is another formulation of the compatibility of Floer data, in terms of these so-called *rotated cylindrical regions.* Let $top(C)$ to be the maximal s coordinate in C ($+\infty$ if C is positive-infinite) and $bottom(C)$ to be the minimal s coordinate in C ($-\infty$ if C is negative-infinite).

Definition 15. *Let $(\theta_1, \ldots, \theta_r)$ be a collection of angles. Define the i*th cumulative angle*, for i from 1 to r, via*

$$\eta_i := \sum_{j=1}^{i} \theta_j; \tag{4.53}$$

or inductively via

$$\begin{aligned} \eta_1 &:= \theta_1 \\ \eta_{i+1} &:= \eta_i + \theta_{i+1}. \end{aligned} \tag{4.54}$$

Also, define $\eta_{-1} = 0$.

Definition 16. *The* (δ-spaced) rotated cylindrical regions *for an r-point angle-decorated cylinder $(C, p_1, \ldots, p_r)$ consist of the following cylindrical ends and finite cylinders:*

- *The* top cylinder

$$\begin{aligned}\epsilon^+ : [0, \max(top(C) - (h_1+\delta), 0)] \times S^1 &\to C \\ (s,t) &\mapsto (\min(s+h_1+\delta, top(C)), t)\end{aligned} \tag{4.55}$$

- *The* bottom cylinder

$$\begin{aligned}\epsilon^- : [\min(bottom(C) - (h_r-\delta), 0), 0] \times S^1 &\to C \\ (s,t) &\mapsto (\max(s-(h_r-\delta), bottom(C)), t+\eta_r).\end{aligned}$$

- *For any* $1 \leq i \leq r-1$ *satisfying* $h_{i+1} - h_i > 3\delta$*, the* ith thin part

$$\begin{aligned}\epsilon_i : [h_{i+1}+\delta, h_i - \delta] \times S^1 &\to C \\ (s,t) &\mapsto (s, t+\eta_i)\end{aligned} \tag{4.56}$$

Note that a given r-point angle-decorated cylinder may not contain the ith thin part, for a given $i \in [1, r-1]$, and indeed may not contain any thin parts. If δ is implicit, we simply refer to above as the rotated cylinder regions.

Now, fixing an δ, it is easy to give a necessary (but not sufficient!) condition for compatibility with gluing:

Definition 17. *We say a Floer datum (K_C, J_C) is is δ-*adapted *to $(C, p_1, \ldots, p_r, K_t, J_t)$ if for any rotated cylindrical region $\epsilon : C' \to C$ associated to $(C, p_1, \ldots, p_r)$ and δ, we have that*

$$\epsilon^*(K_C, J_C) = (K_t, J_t). \tag{4.57}$$

(strictly speaking, this condition is only necessary for sufficiently large finite cylinders). If we were to express this condition without using the rotation already built into the cylindrical region maps, we would arrive at the following, which most closely matches [**BO**].

Definition 18. *Let $(C, p_1, \ldots, p_r)$ be an r-point angle-decorated cylinder, and fix a pair (K_t, J_t) of a time-dependent Hamiltonian and almost complex structure. Let K_C and J_C be a C dependent Hamiltonian and almost complex structure. For a positive constant δ, we say that (K_C, J_C) is* δ-adapted to $(C, p_1, \ldots, p_r, K_t, J_t)$ *if, at a position $z = (s,t)$*

$$\begin{aligned}&(H_z, J_z) = (H_t, J_t) \quad \text{for } s > p_1 + \delta; \\ &(H_z, J_z) = \eta_r^*(H_t, J_t) = (H_{t+\eta_r}, J_{t+\eta_r}) \quad \text{for } s < p_r - \delta; \\ &(H_z, J_z) = \eta_i^*(H_z, J_z) \text{ if } h_{i+1} - h_i > 3\delta \text{ and } s \in [h_{i+1}+\delta, h_i - \delta]\end{aligned} \tag{4.58}$$

4.4. The circle action on the interior. Let us further now suppose that H is chosen to be C^2-small and Morse in the compact region of $\bar{M}$, with perturbation F_t equal to zero in this region. Then it is well known that the symplectic co-chain complex $SC^*(H_t)$ contains a copy of the Morse complex of $H|_{\bar{M}}$ as a subcomplex; the content of this statement

An easy action argument, along with a specific choice of Hamiltonian, shows that

Lemma 11. *There exists a choice of Floer data for the homotopy S^1-action so that $C_{Morse}(H)$ becomes a strictly trivial S^1 subcomplex; meaning that the various operators δ_k, $k \geq 1$, associated to the S^1 action strictly vanish on the subcomplex. Equivalentaly, for this choice of data, on the subcomplex $C_{Morse}(H)$, the A_∞ module action of $C_{-*}(S^1)$ factors through the augmentation map $C_{-*}(S^1) \to \mathbf{k}$.*

Proof. For action reasons, any Floer trajectory with asymptotics at two generators in $C_{Morse}(H)$ remain in the interior of $\bar{M}$. We can choose the Hamiltonian to be autonomous and C^2-small in this region, under which it is known that any Floer trajectory between Morse critical points is in fact a Morse trajectory. For similar reasons, for any moduli space $\mathcal{M}_{-k}$, we can choose all of the Floer data appearing in the to be C^2-small, autonomous, and Morse in this region—in fact, just equal to H. We conclude that for x, y critical points of H, any element u in the parametrized moduli space $\overline{\mathcal{M}}_{-k}(x,y)$ solves an equation that is independent of the choice of parameter $\vec{p} \in (S^1)^k \times (0,1]^{k-1}$.

Namely, u lives in a family of solutions of dimension at least $2k-1$ (given by varying $\vec{p}$), and hence cannot be rigid. The associated operation δ_k, which counts rigid solutions, is therefore zero. $\square$

For such a choice of Hamiltonian, one therefore sees that

COROLLARY 5. *The inclusion chain map*

$$C_{Morse}(H) \to SC^*(M) \tag{4.59}$$

lifts canonically to a chain map

$$C_{Morse}(H)[[u]] \to SC^-_{S^1}(M) = SC^*(M)[[u]]. \tag{4.60}$$

Namely, one has a map

$$H^*(M)[[u]] \to SH^-_{S^1}(M) \tag{4.61}$$

such that the associated composition

$$H^*(M) \to H^*(M)[[u]] \to SH^-_{S^1}(M) \to SH^*(M) \tag{4.62}$$

coincides with the usual map $H^*(M) \to SH^*(M)$.

5. Cyclic open-closed maps

5.1. Open-closed moduli spaces and operations. In this section (which can be skipped upon first read), we assemble the required notion of a Floer perturbation datum on a family of discs, in order to repeatedly use it in geometric constructions which follow.

Let S be a disc with some boundary marked points $z_1, \ldots, z_k$ marked as inputs and an interior marked point p removed, marked as either positive or negative (we consider both). We also equip the interior marked point p with an *asymptotic marker*, that is a half-line $\tau_p \in T_pS$ (or equivalently an element of the unit tangent bundle, defined with respect to some metric). Call any such $S = (S, z_1, \ldots, z_k, p, \tau_p)$ an *open-closed framed disc.*

In addition to the notation for semi-infinite strips (3.39) - (3.40), we use the following notation to refer to the positive and negative semi-infinite cylinder:

$$A_+ := [0, \infty) \times S^1 \tag{5.1}$$

$$A_- := (-\infty, 0] \times S^1 \tag{5.2}$$

DEFINITION 19 (Floer datum). *A* Floer datum $\mathbf{F}_S$ *on a stable open-closed framed disc S consists of the following choices on each component:*

(1) A collection of strip-like and cylindrical ends $\mathfrak{S}$; *that is maps*

$$\epsilon_k^{\pm} : Z_\pm \to S$$
$$\delta_j^{\pm} : A_\pm \to S$$

all with disjoint image in S. These should be chosen so that positive strips and cylinders map to neighborhoods of positively-labeled boundary marked points and interior marked points respectively, and similarly for negative marked points. The cylindrical ends further should be compatible with the asymptotic marker at the given marked point, that is

$$\lim_{s \to \pm\infty} \delta^{\pm}(s, 1) = \tau_p. \tag{5.3}$$

(2) For each strip-like end or cylindrical end $\epsilon_k^{\pm}$, $\delta_j^{\pm}$, a real number, w_k or ν_j, called a weight.

(3) closed 1-form*: a one-form α_S satisfying $d\alpha_S = 0$, and $(\alpha_S)|_{\partial S} = 0$, restricting to wdt on a strip-like or cylindrical end with weight w.*

$$(\epsilon_k^{\pm})^* \alpha_S = w_k dt, \quad (\delta_j^{\pm})\alpha_S = \nu_j dt$$

(4) A main Hamiltonian $H_S : S \to \mathcal{H}(M)$*, restricting to* $\frac{H}{w^2}\psi^w$ *on a strip-like or cylindrical end with weight w:*

$$(\epsilon_k^{\pm})^* H_S = \frac{H}{w_k^2} \circ \psi^{w_k}, \quad (\delta_j^{\pm})^* H_S = \frac{H}{\nu_j^2} \circ \psi^{\nu_j}$$

(5) A perturbation Hamiltonian F_S*, zero on each strip-like end, satisfying*

- *on the cylindrical ends,*

$$(\delta_j^{\pm})^* F_S = \frac{F_T \circ \psi^{\nu_j}}{\nu_j^2} \tag{5.4}$$

- *There are infinitely many values of the cylindrical coordinate on M such that* F_S *is zero at all values of the cylindrical coordinate.*

(6) A boundary-shifting map $\rho : \partial \bar{S} \backslash \{\textit{marked points}\} \to (0, \infty)$ *satisfying*

$$(\epsilon_k^{\pm})^* \rho = w_k.$$

(7) An almost-complex structure J_S *satisfying*

$$(\epsilon_k^{\pm})^* J_S = (\psi^{w_k})^* J_t.$$
$$(\delta^{\pm})^* J_S = (\psi^{\nu})^* J_t.$$

To explain the perturbation Hamiltonian F_S, recall that in the case of wrapped Fukaya categories, one can choose a *quadratic Hamiltonian* whose time-1 chords between any pair of objects is non-degenerate. However, the time-1 orbits of such Hamiltonians come in S^1-families near ∞. It will be therefore be helpful to introduce Hamiltonian perturbation terms on cylindrical ends of our domain in order to break the S^1 symmetry of orbits.

Remark 37. For monotone Lagrangians in monotone symplectic manifolds, one can safely diregard the choice of boundary-shifting map, weight (or rather, set all weights to 1), and perturbation Hamiltonian. We also adopt the convention that the Liouville flow $\psi^t \cong id$ for all t.

Definition 20. *A pair of Floer data* $\mathbf{F}_S^1$, $\mathbf{F}_S^2$ *on S are said to be* conformally equivalent *if for some constant K,*

- *the strip-like and cylindrical ends coincide,*
- $\alpha_1 = K\alpha_2$,
- $\rho_1 = K\rho_2$
- $H_1 = \frac{H_2}{K} \circ \psi^K$*, and*
- $J_1 = (\psi^K)^* J_2$.
- $F_1 = \frac{F_2}{K} \circ \psi^K$

Remark 38. The assumption that F_S vanishes at infinitely many levels is non-essential and can be replaced by a weak monotonicity condition,

$$\partial_s (\delta_j^{\pm})^* F_S \leq 0. \tag{5.5}$$

This can be used to define a larger class of operations than we will do here, at the expense of a more involved proof of compactness of resulting moduli spaces; see [**G3**, Appendix B].

Definition 21. *A* Lagrangian labeling *on a stable open-closed framed disc is an assignment of an object of* $\mathcal{W}$ *to each connected boundary component of* $\partial\bar{S} - \{\textit{boundary marked points}\}$*. We denote a Lagrangian labeling by* $\{L_0, \dots, L_{d-1}\}$*, where* L_i *corresponds to the component between* z_i *and* $z_{i+1-\mathrm{mod}d}$.

Definition 22. *An* admissible collection of asymptotics *for stable open-closed framed disc with Lagrangian labeling is a collection* $\{x_1, \dots, x_d; y\}$ *of chords* $x_i \in \chi(L_{i-1}, L_{i-\mathrm{mod}d})$ *and an orbit* $y \in \mathcal{O}$.

Definition 23. *Given a stable open-closed framed disc S equipped with a Floer datum F_S, a Lagrangian labeling $\{L_0, \ldots, L_{d-1}$, and asymptotics $\{x_1, \ldots, x_d; y\}$, a map $u : S \to M$ satisfies* Floer's equation for F_S with boundary and asymptotics $\{L_0, \ldots, L_{d-1}\}, \{x_1, \ldots, x_d; y\}$ *if*

$$(du - X \otimes \alpha)^{0,1} = 0 \text{ using the Floer data given by } F_S\} \tag{5.6}$$

and u satisfies

$$\begin{cases} u(z) \in \psi^{\rho_S(z)} L_i & \text{if } z \in \partial S \text{ lies between } z_i \text{ and } z_{i+1 \bmod d} \\ \lim_{s \to \pm\infty} u \circ \epsilon^k(s, \cdot) = x_k \\ \lim_{s \to -\infty} u \circ \delta(s, \cdot) = y \end{cases}. \tag{5.7}$$

5.2. Non-unital open-closed maps. We begin by constructing a variant of the open-closed map of [**A**] with source the non-unital Hochschild complex of (3.15), which we call the *non-unital open-closed map*

$$\mathcal{OC}^{nu} : \mathrm{CH}^{nu}_{*-n}(\mathcal{F}, \mathcal{F}) \longrightarrow CF^*(M). \tag{5.8}$$

This map actually has a straightforward explanation from the perspective of Remark 29. Namely, we need to define a pair of maps

$$\check{\mathcal{OC}} \oplus \hat{\mathcal{OC}} : \mathrm{CH}_*(\mathcal{F}, \mathcal{F}) \oplus \mathrm{CH}_*(\mathcal{F}, \mathcal{F})[1] \to CF^*(M) \tag{5.9}$$

giving the left and right component of the map

$$\begin{aligned} \mathcal{OC}^{nu} : \mathrm{CH}^{nu}_{*-n}(\mathcal{F}, \mathcal{F}) &\longrightarrow SC(M), \\ (x, y) &\longmapsto \check{\mathcal{OC}}(x) + \hat{\mathcal{OC}}(y). \end{aligned} \tag{5.10}$$

Since the left (check) factor is a equal to the usual cyclic bar complex for Hochschild homology, $\check{\mathcal{OC}}$ will be defined exactly as the map $\mathcal{OC}$ in [**A**] (This will be briefly recalled below), and the new content is the map $\hat{\mathcal{OC}}$. We will define $\hat{\mathcal{OC}}$ below (and recall the definition of $\check{\mathcal{OC}}$ and prove, extending [**A**] that

Lemma 12. *$\mathcal{OC}^{nu}$ is a chain map.*

Following [**A**], we refer to the map $\check{\mathcal{OC}}$ viewed as a map from $\mathrm{CH}_*(\mathcal{F}, \mathcal{F})$ (the left factor of $\mathrm{CH}^{nu}_*(\mathcal{F}, \mathcal{F})$) as simply $\mathcal{OC}$. It follows from our construction that, assuming Lemma 12,

Corollary 6. *As homology level maps, $[\mathcal{OC}^{nu}] = [\mathcal{OC}]$.*

Proof. By construction, the chain level map $\mathcal{OC}$ factors as

$$\mathrm{CH}_{*-n}(\mathcal{F}, \mathcal{F}) \subset \mathrm{CH}^{nu}_{*-n}(\mathcal{F}, \mathcal{F}) \stackrel{\mathcal{OC}^{nu}}{\longrightarrow} CF^*(M). \tag{5.11}$$

The first inclusion is a quasi-isomorphism by Lemma 3, since $\mathcal{F}$ is known to be cohomologically unital. $\square$

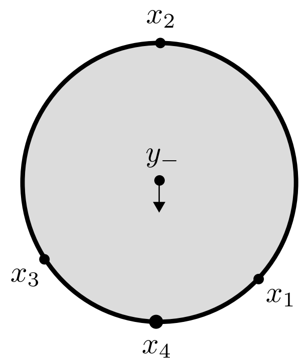
The moduli space controlling the operation $\check{\mathcal{OC}}$, denoted

$$\overline{\check{\mathcal{R}}}^1_d \tag{5.12}$$

is the (Deligne-Mumford compactification of the) abstract moduli space of discs with d boundary positive punctures $z_1, \ldots, z_d$ labeled in counterclockwise order and 1 interior negative puncture p_{out}, with an asymptotic marker τ_{out} at p_{out} pointing towards z_d. The space (5.12) has a manifold with corners structure, with boundary strata described in [**A**, §C.3] (there, the space is called $\overline{\mathcal{R}}^1_d$)–in short, codimension one strata consist of disc bubbles containing any cyclic subseqeuence of k inputs attached to an element of $\check{\mathcal{R}}^1_{d-k+1}$ at the relative position of this cyclic subsequence. Orient the top stratum $\mathcal{R}^1_d$ by trivializing it, sending $[S]$ to the unit disc representative S with z_d and p_{out} fixed at 1 and 0, and taking the orientation induced by the (angular) positions of the remaining marked points:

$$-\, dz_1 \wedge \cdots \wedge dz_{d-1}. \tag{5.13}$$

FIGURE 1. A representative of an element of the moduli space $\check{\mathcal{R}}^1_4$ with special points at 0 (output), $-i$.



The moduli space controlling the new map $\hat{\mathcal{OC}}$ is nearly identical to $\check{\mathcal{R}}^1_d$, but there additional freedom in the direction of the asymptotic marker at the interior puncture p_{out}. The top (open) stratum is easiest to define: let

$$\mathcal{R}^{1,free}_d \tag{5.14}$$

be the moduli space of discs with d positive boundary punctures and one interior negative puncture as in $\check{\mathcal{R}}^1_d$, but with the asymptotic marker τ_{out} pointing anywhere between z_1 and z_d.

REMARK 39. There is a delicate point in naively compactifying $\mathcal{R}^{1,free}_d$: on any formerly codimension 1 stratum in which z_1 and z_d bubble off, the position of τ_{out} becomes fixed too, and so the relevant stratum actually should have codimension 2 (and hence does not contribute to the codimension-1 boundary equation for $\hat{\mathcal{OC}}$. Moreoever, there is no nice corner chart near this stratum). For technical convenience, we pass to an alternate, larger (blown-up) model for the compactification in which these strata have codimension 1 but consist of degenerate contributions.

In light of Remark 39, we use (5.14) as motivation and instead define

$$\hat{\mathcal{R}}^1_d \tag{5.15}$$

to be the abstract moduli space of discs with $d+1$ boundary punctures z_f, z_1, $\ldots$, z_d and an interior puncture z_{out} with asymptotic marker τ_{out} pointing towards the boundary point z_f, modulo automorphism. We mark z_f as "auxiliary," but otherwise the space is abstractly isomorphic to $\check{\mathcal{R}}^1_{d+1}$. Identifying $\hat{\mathcal{R}}^1_d$ with the space of unit discs with z_{out} and z_f fixed at 1 and 0, the remaining (angular) positions of $z_1, \ldots, z_d$ determine an orientation

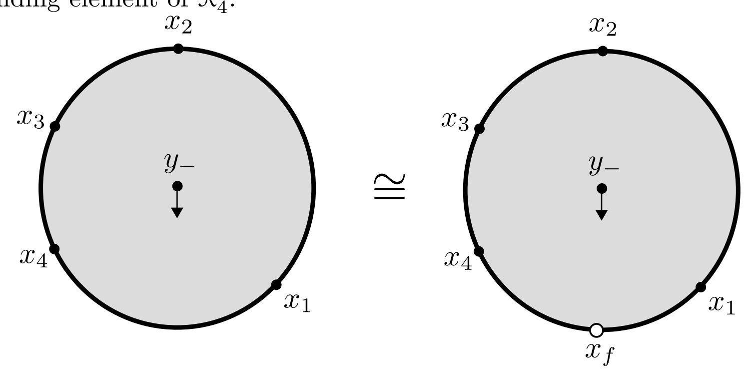
$$-\, dz_1 \wedge \cdots \wedge dz_d. \tag{5.16}$$

The *forgetful map*

$$\pi_f : \hat{\mathcal{R}}^1_d \to \mathcal{R}^{1,free}_d \tag{5.17}$$

puts back in the point z_f and forgets it. Since the point z_f is recoverable from the direction of the asymptotic marker at z_{out},

FIGURE 2. A representative of an element of the moduli space $\mathcal{R}^1_{4,free}$ and the corresponding element of $\hat{\mathcal{R}}^1_4$.



LEMMA 13. *π_f is a diffeomorphism.*

The perspective of the former space (5.15) gives us a model for the compactification

$$\overline{\mathcal{R}}^{1,free}_d \tag{5.18}$$

as the ordinary Deligne-Mumford compactification

$$\overline{\hat{\mathcal{R}}}^1_d. \tag{5.19}$$

As a manifold with corners, (5.19) is equal to the compactification $\overline{\check{\mathcal{R}}}^1_{d+1}$, except from the point of view of assigning Floer datum, as we will be forgetting the point z_f instead of fixing asymptotics for it. It is convenient therefore to name components of strata containing z_f differently. At any stratum:

- we treat the main component (containing z_{out} and k boundary marked points) as belonging to $\overline{\hat{\mathcal{R}}}^1_{k-1}$ if it contains z_f and $\overline{\check{\mathcal{R}}}^1_k$ otherwise; and
- If the ith boundary marked point of any non-main component was z_f, we view it as an element of $\mathcal{R}^{k,f_i}$, the space of discs with 1 output and k input marked points removed from the boundary, with the ith point marked as "forgotten," constructed in Appendix A.2.
- We treat any other non-main component as belonging to $\mathcal{R}^k$ as usual.

Thus, the codimension-1 boundary of the Deligne-Mumford compactification is covered by the natural inclusions of the following strata

$$\overline{\mathcal{R}}^m \times_i \overline{\hat{\mathcal{R}}}^1_{d-m+1} \quad 1 \leq i < d-m+1 \tag{5.20}$$

$$\overline{\mathcal{R}}^{m,f_k} \times_{d-m+1} \overline{\check{\mathcal{R}}}^1_{d-m+1} \quad 1 \leq j \leq m, \ 1 \leq k \leq m \tag{5.21}$$

where the notation $\times_j$ means that the output of the first component is identified with the jth boundary input of the second.

The forgetful map π_f extends to a map $\overline{\pi}_f$ from the compactification $\overline{\hat{\mathcal{R}}}^1_d$ as follows: we call a component T of a representative S of $\overline{\mathcal{R}}^1_d$ the *main component* if it contains the interior marked point, and the *secondary component* if its output is attached to the main component. Then, $\overline{\pi}_f$ puts the auxiliary point z_f back in, eliminates any component which is not main or secondary which has only one non-auxiliary marked point p, and labels the positive marked point below this component by p. Given a representative S of $\overline{\hat{\mathcal{R}}}^1_d$, we call $\overline{\pi}_f(S)$ the *associated reduced surface.* We will study maps from the associated reduced surfaces $\overline{\pi}_f(S)$, parametrized by S. To this end, we define a *Floer datum* on a stable disc S in $\overline{\hat{\mathcal{R}}}^1_d$ to consist of a Floer datum for the underlying reduced surface $\overline{\pi}_f(S)$.

As a prerequesite to the forthcoming inductive choices, in Appendix A.2 we describe an inductive construction of Floer data for the moduli space of discs with a forgotten point $\mathcal{R}^{d,f_i}$.

Definition 24. *A* universal and consistent choice of Floer data for the non-unital open-closed map *is inductive set of choices* $(\mathbf{D}_{\check{\mathcal{OC}}}, \mathbf{D}_{\hat{\mathcal{OC}}})$ *for Floer data for each* $d \geq 1$ *and every representative* $S \in \overline{\check{\mathcal{R}}}^1_d$, $T \in \overline{\hat{\mathcal{R}}}^1_d$, *varying smoothly over each of these moduli spaces, whose restriction to a boundary stratum is conformally equivalent to a product of Floer data coming from lower dimensional moduli spaces. Near the boundary strata, with regards to standard previously chosen gluing coordinates, this choice agrees to infinite order with the Floer data obtained by gluing. In particular, as described in Appendix A.2,*

$$\text{the choice of Floer datum on strata containing } \mathcal{R}^{d,f_i} \text{ components} \\ \text{should be constant along fibers of the forgetful map } \mathcal{R}^{d,f_i} \to \mathcal{R}^{d-1}. \tag{5.22}$$

Lemma 14. *Universal and consistent choices of Floer data for the non-unital open-closed map exist.*

Proof. As usual, this is an inductive argument in d. For a given d, one first chooses a Floer datum extending the (well-defined) choices imposed on boundary strata by consistency, first for $\overline{\check{\mathcal{R}}}^1_d$ and then for $\overline{\hat{\mathcal{R}}}^1_d$ (as the latter space contains the former space in its boundary strata). Contractibility of the space of choices ensures that at each stage compatible choices exist.

□

Fixing a universal and consistent choice and Floer data, we obtain, for any d-tuple of Lagrangians $L_0, \ldots, L_{d-1}$, and asympotic conditions

$$\begin{aligned} \vec{x} &= (x_d, \ldots, x_1),\ x_i \in \chi(L_{i-1}, L_{i-\text{mod}d}) \\ y_{out} &\in \mathcal{O} \end{aligned} \tag{5.23}$$

a pair of moduli spaces

$$\check{\mathcal{R}}^1_d(y_{out}; \vec{x}) \tag{5.24}$$

$$\hat{\mathcal{R}}^1_d(y_{out}; \vec{x}), \tag{5.25}$$

of parametrized families of solutions to Floer's equation

$$\{(S,u) | S \in \mathcal{R}^1_d : u : S \to M, (du - X \otimes \alpha)^{0,1} = 0 \text{ using the Floer data given by } \mathbf{D}_{\check{\mathcal{OC}}}(S)\} \tag{5.26}$$

$$\{(S,u) | S \in \hat{\mathcal{R}}^1_d, u : \pi_f(S) \to M | (du - X \otimes \alpha)^{0,1} = 0 \text{ using the Floer datum given by } \mathbf{D}_{\hat{\mathcal{OC}}}(S)\} \tag{5.27}$$

satisfying asymptotic and moving boundary conditions (in either case)

$$\begin{cases} u(z) \in \psi^{\rho_S(z)} L_i & \text{if } z \in \partial S \text{ lies between } z_i \text{ and } z_{i+1 \text{ mod } d} \\ \lim_{s \to \pm\infty} u \circ \epsilon^k(s, \cdot) = x_k & \\ \lim_{s \to -\infty} u \circ \delta(s, \cdot) = y & \end{cases}. \tag{5.28}$$

Construct the usual Gromov-type bordification $\overline{\check{\mathcal{R}}}^1_d(y_{out}; \vec{x})$, $\overline{\hat{\mathcal{R}}}^1_d(y_{out}; \vec{x})$ by allowing semi-stable breakings, as well as maps from strata corresponding to the boundary strata of $\overline{\check{\mathcal{R}}}^1_d$ and $\overline{\hat{\mathcal{R}}}^1_d$.

Lemma 15 (Transversality, index calculations, and compactness). *For generic choices of Floer data, the Gromov-type compactifications*

$$\overline{\check{\mathcal{R}}}^1_d(y_{out}; \vec{x}); \tag{5.29}$$

$$\overline{\hat{\mathcal{R}}}^1_d(y_{out}; \vec{x}), \tag{5.30}$$

are smooth compact manifolds of the following dimensions respectively:

$$\deg(y_{out}) - n + d - 1 - \sum_{k=1}^{d} \deg(x_k); \tag{5.31}$$

$$\deg(y_{out}) - n + d - \sum_{k=1}^{d} \deg(x_k). \tag{5.32}$$

For $\deg(y_{out}) = d-n-1+\sum_{k=1}^{d}\deg(x_k)$ or $\deg(y_{out}) = d-n+\sum_{k=1}^{d}\deg(x_k)$ respectively, each element of $u \in \overline{\check{\mathcal{R}}}_d^1(y_{out};\vec{x})$ or $u \in \overline{\hat{\mathcal{R}}}_d^1(y_{out};\vec{x})$ respectively is rigid and gives, using the fixed orientations of moduli spaces of domains (5.13)-(5.16) and [**A**, Lemma C.4], isomorphisms of orientation lines

$$(\check{\mathcal{R}}_d^1)_u : o_{x_d} \otimes \cdots \otimes o_{x_1} \to o_{y_{out}} \tag{5.33}$$

$$(\hat{\mathcal{R}}_d^1)_u : o_{x_d} \otimes \cdots \otimes o_{x_1} \to o_{y_{out}}. \tag{5.34}$$

These isomorphisms in turn define the $|o_{y_{out}}|_{\mathbf{k}}$ component of the check and hat components of the non-unital open-closed map with d inputs in the lines $|o_{x_d}|_{\mathbf{k}}, \ldots, |o_{x_1}|_{\mathbf{k}}$, up to a sign twist:

$$\begin{aligned} &\check{\mathcal{OC}}_d([x_d]\otimes\cdot\otimes[x_1]) := \\ &\qquad \sum_{\deg(y)=n-d+1+\sum\deg(x_i)} \ \sum_{u\in\overline{\check{\mathcal{R}}}_1^d(y;x_d,\ldots,x_1)} (-1)^{\check{\star}_d}(\check{\mathcal{R}}_d^1)_u([x_d],\ldots,[x_1]), \\ &\check{\star}_d := \deg(x_d) + \sum_{k=1}^{d} k\deg(x_k) \end{aligned} \tag{5.35}$$

$$\begin{aligned} &\hat{\mathcal{OC}}_d([x_d],\ldots,[x_1]) := \sum_{\deg(y_{out})=d-n+\sum\deg(x_k)} \ \sum_{u\in\overline{\hat{\mathcal{R}}}_d^1(y_{out};\vec{x})} (-1)^{\hat{\star}_d}(\hat{\mathcal{R}}_d^1)_u([x_d],\ldots,[x_1]), \\ &\hat{\star}_d := \sum_{i=1}^{d} i\cdot\deg(x_i). \end{aligned} \tag{5.36}$$

By analyzing the boundary of one-dimensional components of the moduli spaces $\overline{\check{\mathcal{R}}}_d^1(y_{out};\vec{x})$, the consistency condition imposed on Floer data, and a sign analysis, it was proven in [**A**] that

Lemma 16 ([**A**], Lemma 5.4). *$\mathcal{OC} := \check{\mathcal{OC}}$ is a chain map of degree n; that is $(-1)^n d_{CF}\circ\check{\mathcal{OC}} = \check{\mathcal{OC}}\circ b$.*

Similarly, we prove the following, completing the proof of Lemma 12:

Lemma 17. *The following equation holds:*

$$(-1)^n d_{CF}\circ\hat{\mathcal{OC}} = \check{\mathcal{OC}}\circ d_{\wedge\vee} + \hat{\mathcal{OC}}\circ b' \tag{5.37}$$

Proof. The consistency condition imposed on Floer data implies that the codimension 1 boundary of the Gromov bordification $\overline{\check{\mathcal{R}}}_d^1(y;\vec{x})$ is covered by the images of the natural inclusions of the moduli spaces of maps coming from the boundary strata (5.20), (5.21) along with semi-stable breakings

$$\overline{\hat{\mathcal{R}}}_d^1(y_1;\vec{x}) \times \overline{\mathcal{M}}(y_{out};y_1) \to \partial\overline{\check{\mathcal{R}}}_d^1(y_{out};\vec{x}) \tag{5.38}$$

$$\overline{\mathcal{R}}^1(x_1;x) \times \overline{\hat{\mathcal{R}}}_d^1(y_{out};\tilde{\vec{x}}) \to \partial\overline{\check{\mathcal{R}}}_d^1(y_{out};\vec{x}),. \tag{5.39}$$

Let $\mu^{d,i}$ be the operation associated to the space of discs with ith point marked as forgotten $\mathcal{R}^{d,f_i}$, which is described in detail in Appendix A.2. $\mu^{d,i}$ takes a composable sequence of $d-1$ inputs, separated into an $i-1$ tuple and a $d-i$ tuple; in line with Remark 29 we will use the suggestive notation

$$\mu^d(x_d, \ldots, x_{i+1}, e^+, x_{i-1}, \ldots, x_1) := \mu^{d,i}(x_d, \ldots, x_{i+1}; x_{i-1}, \ldots, x_1). \tag{5.40}$$

[8] Then, up to sign, by the standard codimension-1 boundary principle for Floer-theoretic operations, we have shown that

(5.41)
$$\begin{aligned} 0 =& d_{CF}\hat{\mathcal{OC}}(x_d, \ldots, x_1) - \sum_{i,j} (-1)^{\maltese_1^i} \hat{\mathcal{OC}}(x_d \otimes \cdots \otimes x_{i+j+1} \otimes \mu^j(x_{i+j}, \ldots, x_{i+1}) \otimes x_i \otimes \cdots \otimes x_1) \\ & - \sum_{i,j,k} (-1)^{\sharp_j^k} \check{\mathcal{OC}}(\mu^{j+k+1}(x_j, \ldots, x_1, e^+, x_d, \ldots, x_{d-k+1}) \otimes x_{d-k} \otimes \cdots \otimes x_{j+1}). \end{aligned}$$

with desired signs

$$\maltese_m^n = \sum_{j=m}^{n} ||x_i|| \tag{5.42}$$

$$\sharp_j^k = \maltese_1^j \maltese_{j+1}^d + \maltese_{j+1}^d + 1 \tag{5.43}$$

However, as shown in Appendix A.2,

$$\mu^{j+k+1}(x_j, \ldots, x_1, e^+, x_d, \ldots, x_{d-k+1}) = \begin{cases} x_1 & j=1,\ k=0 \\ (-1)^{|x_d|} x_d & j=0, k=1 \\ 0 & \text{otherwise} \end{cases} \tag{5.44}$$

(in this manner, e^+, though a formal element, behaves as a strict unit). So if (5.41) held, it would follow that

$$\begin{aligned} d_{CF} \circ \hat{\mathcal{OC}}(x_d \otimes \cdots \otimes x_1) &= (-1)^{||x_1||\maltese_2^d + \maltese_2^d + 1} \mathcal{OC}(x_1 \otimes x_d \otimes \cdots \otimes x_2) \\ &\quad + (-1)^{|x_d| + \maltese_1^d + 1} \check{\mathcal{OC}}(x_d \otimes \cdots \otimes x_1) + \hat{\mathcal{OC}} \circ b'(x_d \otimes \cdots \otimes x_1) \\ &= \check{\mathcal{OC}}((-1)^{\maltese_1^d + ||x_d||}(1-t)(x_d \otimes \cdots \otimes x_1)) + \hat{\mathcal{OC}} \circ b'(x_d \otimes \cdots \otimes x_1). \\ &= \left(\check{\mathcal{OC}} \circ d_{\wedge\vee} + \hat{\mathcal{OC}} \circ b'\right) \circ (x_d \otimes \cdots \otimes x_1). \end{aligned} \tag{5.45}$$

So we are done if we establish the signs are exactly (5.42)-(5.43).

Using the notation

$$\mathcal{OC}(e^+ \otimes x_d \otimes \cdots \otimes x_1) := \hat{\mathcal{OC}}(x_d \otimes \cdots \otimes x_1), \tag{5.46}$$

where again e^+ is simply a formal symbol referring to the position of the auxiliary (forgotten) input point, we observe that the equation (5.41) is exactly the equation for $\mathcal{OC}$ being a chain map on inputs of the form $(e^+ \otimes x_d \otimes \cdots \otimes x_1)$ (where we treat an "e^+" input as an auxiliary unconstrained point on our domain). The sign verification therefore follows from that of $\check{\mathcal{OC}}$ being a chain map (in [**A**, Lemma 5.4]), for we have used identical orientations on the abstract moduli space $\hat{\mathcal{R}}_d^1$ as on $\check{\mathcal{R}}_{d+1}^1$, and on $\mathcal{R}^{d,f_i}$ as on $\mathcal{R}^d$, and we can even insert a formal degree zero orientation line o_{e^+} into the procedure for orienting moduli spaces of open-closed maps (see [**A**, §C.6]), corresponding to the marked point x_f. Note that o_{e^+}, being of degree zero, commutes with everything, and is just used as a placeholder as if we had an asymptotic condition at x_f. □

Proof of Lemma 12. Given that $\check{\mathcal{OC}}$ is already known to be a chain map by [**A**, Lemma 5.4], repeated as Lemma 16 above, the new part to check is that the $d_{CF} \circ \hat{\mathcal{OC}} = \check{\mathcal{OC}} d_{\wedge\vee} + \hat{\mathcal{OC}} \circ b'$. This is the content of Lemma 17 above. □

[8] In fact, when the Fukaya category is equipped with homotopy units, one can ensure that there is a strict unit element e^+ in each self-hom space, for which μ^k with an e^+ element admits a geometric description as above. See e.g., [**FOOO**] or [**G2**].

5.3. An auxiliary operation. It will be technically convenient to define an auxiliary operation

$$\mathcal{OC}^{S^1} : \mathrm{CH}_{*-n}(\mathcal{F}, \mathcal{F}) \to CH^{*+1}(M) \tag{5.47}$$

from the left factor of the non-unital Hochschild complex to Floer co-chains, in which the asymptotic marker τ_{out} varies freely around the circle. This operation is more easily comparable to the BV operator on Floer cohomology, and moreover, we will show that $\mathcal{OC}^{S^1}$ (and $\hat{\mathcal{OC}}$) can be chosen to satisfy the following crucial identity:

Proposition 7. *There is an equality of chain level operations:*

$$\mathcal{OC}^{S^1} = \hat{\mathcal{OC}} \circ B^{nu}. \tag{5.48}$$

To define (5.47), let

$$\mathcal{R}_d^{S^1} \tag{5.49}$$

be the abstract moduli space of discs with d boundary positive punctures $z_1, \ldots, z_d$ labeled in counterclockwise order and 1 interior negative puncture p_{out}, with an asymptotic marker τ_{out} at p_{out} (or choice of real half line in $T_{p_{out}}D$) which is free to vary. Equivalently,

(5.50) (5.49) is the space of discs with $z_1, \ldots, z_d$ and p_{out} as before, and an extra auxiliary interior marked point p_1 such that, for a representative with (p_{out}, z_1) fixed at $(0, -i)$, $|p_1| = \frac{1}{2}$, and the asymptotic marker τ_{out} points towards p_1.

Abstractly,

$$\mathcal{R}_d^{S^1} = S^1 \times \mathcal{R}_d^1, \tag{5.51}$$

a fact which we use to fix an orientation on the top component of (5.51). The Deligne-Mumford type compactification thus has a simple description

$$\overline{\mathcal{R}_d^{S^1}} = \mathcal{R}_d^1 \times S^1. \tag{5.52}$$

Given an element S of $\mathcal{R}_d^{S^1}$ and a choice of marked point z_i on the boundary of S, we say say that τ_{out} *points at* z_i, if, when S is reparametrized so that z_1 fixed at $-i$ and p_{out} fixed at 0, the vector τ_{out} is tangent to the straight line from p_{out} to z_i. Equivalently, for this representative, p_{out}, p_1, and z_i are collinear. For each i, the locus where τ_{out} points at z_i forms a codimension 1 submanifold, denoted

$$\mathcal{R}_d^{S^1_i}. \tag{5.53}$$

The notion compactifies well; if z_i is not on the main component of (5.51) we say τ_{out} *points at* z_i if it points at the root of the bubble tree z_i is on. This compactified locus $\overline{\mathcal{R}_d^{S^1,i}}$ can be identified on the nose with $\overline{\mathcal{R}}_d^1$ via the map

$$\tau_i : \overline{\mathcal{R}_d^{S^1,i}} \to \overline{\mathcal{R}}_d^1 \tag{5.54}$$

which cyclically permutes the labels of the boundary marked points so that z_i is now labeled z_0.

In a similar fashion, we have an invariant notion of what it means for τ_{out} to point *between* z_i *and* z_{i+1}; this is a codimension 0 submanifold with corners of (5.51), denoted

$$\mathcal{R}_d^{S^1_{i,i+1}}. \tag{5.55}$$

The compactification has some components that are codimension 1 submanifolds with corners of (5.51), when z_i and z_{i+1} both lie on a bubble tree.

Finally, there is a *free* $\mathbb{Z}_d$ action generated by the map

$$\kappa : \overline{\mathcal{R}_d^{S^1}} \to \overline{\mathcal{R}_d^{S^1}} \tag{5.56}$$

which cyclically permutes the labels of the boundary marked points; for concreteness, κ changes the label z_i to z_{i+1} for $i < d$, and z_d to z_1. Note that if, on a given S, τ_{out} points between z_i and z_{i+1}, then on $\kappa(S)$, τ_{out} points between $z_{i+1 \text{ mod d}}$ and $z_{i+2 \text{ mod d}}$.

We now choose Floer perturbation data for this family of moduli spaces; in fact, it will be helpful to re-choose Floer data for the moduli spaces appearing in the non-unital open-closed map to have extra compatibility. To that end, a *BV compatible universal and consistent Floer datum* for the *non-unital open-closed operation* is an inductive choice $(\mathbf{D}_{\mathcal{O}\check{\mathcal{C}}}, \mathbf{D}_{\hat{\mathcal{O}\mathcal{C}}}, \mathbf{D}_{S^1})$ of Floer data where $\mathbf{D}_{\mathcal{O}\check{\mathcal{C}}}$ and $\mathbf{D}_{\hat{\mathcal{O}\mathcal{C}}}$ is a universal and consistent choice of Floer data for the non-unital open-closed map as before, and $\mathbf{D}_{S^1}$ consists of, for each $d \geq 1$ and every representative $S \in \overline{\mathcal{R}_d^{S^1}}$, varying smoothly over the moduli space, whose restriction to a boundary stratum is conformally equivalent to a product of Floer data coming from lower dimensional moduli spaces. Near the boundary strata, with regards to standard previously chosen gluing coordinates, this choice agrees to infinite order with the Floer data obtained by gluing. Moreover, there are two additional inductive constraints:

(5.57) On the codimension-1 loci $\overline{\mathcal{R}_d^{S^1_i}}$ where τ_{out} points at z_i, the Floer datum should agree with the pullback by τ_i of the existing Floer datum for the (check) open-closed map.

(5.58) The Floer datum should be κ-equivariant, where κ is the map (5.56).

Also, there is a final a posteriori constraint on the Floer data for the non-unital open-closed map $\mathbf{D}_{\hat{\mathcal{O}\mathcal{C}}}$; for $S \in \overline{\hat{\mathcal{R}}}^1_d$:

$$\begin{aligned}&\text{the Floer datum on the main component } S_0 \text{ of } \pi_f(S) \text{ should coincide with the}\\ &\text{existing datum chosen on } S_0 \in \mathcal{R}_d^{1,free} \subset \mathcal{R}_d^{S^1}.\end{aligned} \tag{5.59}$$

By an inductive argument as before, universal and consistent choices of Floer data exist (though now we choose the data for $\mathcal{R}_d^{S^1}$ prior to choosing that of $\overline{\hat{\mathcal{R}}}^1_d$).

In fact, the condition (5.59) specifies the Floer datum entirely. We then observe that (5.22) is compatibile with consistency and the condition (5.57).

The second constraint, (5.58) is compatible with the first (5.57) (which is visibly an equivariant condition), and generally does not impose a problem, as the action generated by κ is free. Specifically, one can pick a Floer datum first for surfaces S with τ_{out} pointing between z_d and z_1, and then use equivariance to determine general τ_{out} (there is a fact to check that the Floer datum can be chosen to be just smooth and not piecewise smooth, but this too is straightforward).

Fixing a universal and consistent choice and Floer data, we again obtain, for any d-tuple of Lagrangians $L_0, \ldots, L_{d-1}$, and asympotics $\vec{x} = (x_d, \ldots, x_1)$ $(x_i \in \chi(L_i, L_{i+1 \text{ mod } d}))$, $y_{out} \in \mathcal{O}$, a moduli space

$$\mathcal{R}_d^{S^1}(y_{out}; \vec{x}), \tag{5.60}$$

of parametrized families of solutions to Floer's equation

$$\{(S, u) | S \in \mathcal{R}_d^{S^1}, u : \pi_f(S) \to M | (du - X \otimes \alpha)^{0,1} = 0\} \tag{5.61}$$

satisfying asymptotic and moving boundary conditions

$$\begin{cases} u(z) \in \psi^{\rho_S(z)} L_i & \text{if } z \in \partial S \text{ lies between } z_i \text{ and } z_{i+1 \text{ mod } d} \\ \lim_{s \to \pm\infty} u \circ \epsilon^k(s, \cdot) = x_k & \\ \lim_{s \to -\infty} u \circ \delta(s, \cdot) = y & \end{cases}. \tag{5.62}$$

Generically the Gromov compactification

$$\overline{\mathcal{R}}_d^{S^1}(y_{out}; \vec{x}), \tag{5.63}$$

is a smooth compact manifold of dimension

$$\deg(y_{out}) - n + d - \sum_{k=1}^{d} \deg(x_k). \tag{5.64}$$

When this dimension is 0, i.e., $\deg(y_{out}) = d - n + \sum_{k=1}^{d} \deg(x_k)$, each $u \in \overline{\mathcal{R}}_d^{S^1}(y_{out}; \vec{x})$ is rigid and gives by the orientation from (5.51) and [**A**, Lemma C.4] an isomorphism of orientation lines

$$(\mathcal{R}_d^{S^1})_u : o_{x_d} \otimes \cdots \otimes o_{x_1} \to o_{y_{out}}, \tag{5.65}$$

which defines the $|o_{y_{out}}|_{\mathbf{k}}$ component of the S^1 open-closed map with d inputs in the lines $|o_{x_d}|_{\mathbf{k}}, \ldots, |o_{x_1}|_{\mathbf{k}}$, up to a sign twist:

$$\begin{aligned} \mathcal{OC}^{S^1}([x_d], \ldots, [x_1]) &:= \sum_{\deg(y_{out}) = d-n+\sum \deg(x_k)} \sum_{u \in \overline{\mathcal{R}}_d^{S^1}(y_{out}; \vec{x})} (-1)^{\clubsuit_d} (\mathcal{R}_d^{S^1})_u([x_d], \ldots, [x_1]), \\ \clubsuit_d &= \sum_{i=1}^{d} (i+1) \cdot \deg(x_i) + \deg(x_d) + d - 1 \end{aligned} \tag{5.66}$$

The proof of Proposition 7, which equates $\mathcal{OC}^{S^1}$ with $\mathcal{OC} \circ B^{nu}$, has two steps. First, we decompose the moduli space $\mathcal{R}_d^{S^1}$ into sectors in which τ_{out} points between a pair of adjacent boundary marked points. It will follow that the sum of the corresponding "sector operations" is exactly $\mathcal{OC}^{S^1}$. The sector operations in turn can be compared to $\hat{\mathcal{OC}}$ via cyclically permuting inputs and an orientation analysis.

We begin by defining the relevant sector operations: For $i \in \mathbb{Z}/(d+1)\mathbb{Z}$, define

$$\hat{\mathcal{R}}^1_{d,\tau_i} \tag{5.67}$$

to be the abstract moduli space of discs with $d+1$ boundary punctures $z_1, \ldots, z_i, z_f, z_{i+1}, \ldots, z_d$ arranged in clockwise order and interior puncture z_{out} with asymptotic marker pointing towards the boundary point z_f, which is also marked as "auxiliary." There is a bijection

$$\tau_i : \hat{\mathcal{R}}^1_{d,\tau_i} \simeq \hat{\mathcal{R}}^1_d \tag{5.68}$$

given by cyclically permuting labels, which induces a model for the compactification $\overline{\hat{\mathcal{R}}^1_{d,\tau_i}}$. However, we will use a different orientation than the one induced by pullback: on a slice with fixed position of z_d and z_{out}, we take the volume form

$$dz_1 \wedge \cdots \wedge dz_{d-1} \wedge dz_f. \tag{5.69}$$

By construction, the induced "forgetful map"

$$\pi_f^i : \hat{\mathcal{R}}^1_{d,\tau_i} \to \mathcal{R}^{S^1_{i,i+1}}, \tag{5.70}$$

is an oriented diffeomorphism that extends to a map between compactifications (note as before that strictly speaking this map does not forget any information, at least on the open locus).

Remark 40. In the case $i = 0$, note that this orientation agrees with the previously chosen orientation (5.16) on $\hat{\mathcal{R}}^1_d$. To see this, note that we previously defined the orientation on $\hat{\mathcal{R}}^1_d$ in terms of a different slice of the group action. To compare the forms $dz_1 \wedge \cdots \wedge dz_{d-1} \wedge dz_f$ (coming from the slice with fixed z_d and z_{out}) and $-dz_1 \wedge \cdots \wedge dz_d$ (coming from the slice with fixed z_f and z_{out}), note that either orientation is induced by the following procedure:

- fix an orientation on the space of discs as above with fixed position of z_{out} (but not z_f or z_d): we shall fix the canonical orientation $dz_1 \wedge \cdots \wedge dz_d \wedge dz_f$;
- fix a choice of trivalizing vector field for the remaining S^1 action on this space of discs with fixed z_{out}: we shall fix $S = (-\partial_{z_f} - \partial_{z_1} - \cdots - \partial_{z_d})$; and

- fix a convention for contracting orientation forms along slices of the action: to determine the orientation on a slice of an S^1 action, we will contract the orientation on the original space on the right by the trivializing vector field.

Moreoever, this data induces an orientation on the quotient by the S^1 action, and also an oriented isomorphism between the induced orientation on any slice and that of the quotient. It follows that on the quotient, the orientation $-dz_1 \wedge \cdots \wedge dz_d$ (from the slice where z_f is fixed), and the orientation $dz_1 \wedge \cdots \wedge dz_{d-1} \wedge dz_f$ (from the slice where z_d is fixed) agree. We conclude these two orientations agree. The author thanks Nick Sheridan for relevant discussions about orientations of moduli spaces.

Choose as a Floer datum for $\overline{\mathcal{R}^1_{d,\tau_i}}$ the pulled back Floer datum from $\overline{\hat{\mathcal{R}}^1_d}$ via (5.68); it automatically then exists and is universal and consistent as desired. Fixing this choice, for any d-tuple of Lagrangians $L_0, \ldots, L_{d-1}$, and asympotic conditions $\vec{x} = (x_d, \ldots, x_1)$, $x_i \in \chi(L_i, L_{i+1 \text{ mod d}})$, $y_{out} \in \mathcal{O}$ we obtain a moduli space

$$\mathcal{R}^1_{d,\tau_i}(y_{out}; \vec{x}) = \hat{\mathcal{R}}^1_d(y_{out}; (x_{i-1}, \ldots, x_1, x_d, \ldots, x_i)) \tag{5.71}$$

of parametrized families of solutions to Floer's equation

$$\{(S, u) | S \in \hat{\mathcal{R}}^1_d, u : \pi_f(S) \to M | (du - X \otimes \alpha)^{0,1} = 0 \text{ using the Floer datum for } \pi_f(S)\} \tag{5.72}$$

satisfying asymptotic and moving boundary conditions

$$\begin{cases} u(z) \in \psi^{\rho_S(z)} L_i & \text{if } z \in \partial S \text{ lies between } z_i \text{ and } z_{i+1 \text{ mod } d} \\ \lim_{s \to \pm\infty} u \circ \epsilon^k(s, \cdot) = x_k & \\ \lim_{s \to -\infty} u \circ \delta(s, \cdot) = y & \end{cases} \tag{5.73}$$

and its Gromov-type compactification

$$\overline{\mathcal{R}}^1_{d,\tau_i}(y_{out}; \vec{x}) := \overline{\hat{\mathcal{R}}^1_d}(y_{out}; (x_i, \ldots, x_1, x_d, \ldots, x_{i+1})), \tag{5.74}$$

which is a smooth compact manifold of dimension $\deg(y_{out}) - n + d - \sum_{j=0}^d \deg(x_j)$.

For $\deg(y_{out}) = d - n + \sum_{j=1}^d \deg(x_j)$, each element $u \in \overline{\mathcal{R}}^1_{d,\tau_i}(y_{out}; \vec{x})$ is rigid and gives by (5.69) and [**A**, Lemma C.4] an isomorphism of orientation lines

$$(\mathcal{R}^1_{d,\tau_i})_u : o_{x_d} \otimes \cdots \otimes o_{x_1} \to o_{y_{out}}, \tag{5.75}$$

which defines the $|o_{y_{out}}|_{\mathbf{k}}$ component of an operation $\hat{\mathcal{OC}}_{d,\tau_i}$ with d inputs in the lines $|o_{x_d}|_{\mathbf{k}}, \ldots, |o_{x_1}|_{\mathbf{k}}$, up to the following sign twist:

$$\begin{gathered} \hat{\mathcal{OC}}_{d,\tau_i}([x_d], \ldots, [x_1]) := \sum_{\deg(y_{out}) = d - n + \sum \deg(x_k)} \sum_{u \in \overline{\mathcal{R}}^1_{d,\tau_i}(y_{out}; \vec{x})} (-1)^{\clubsuit_d} (\hat{\mathcal{R}}^1_{d,\tau_i})_u([x_d], \ldots, [x_1]), \\ \clubsuit_d = \sum_{i=1}^d (i+1) \cdot \deg(x_i) + \deg(x_d) + d - 1. \end{gathered} \tag{5.76}$$

Lemma 18. *As chain level operations,*

$$\mathcal{OC}^{S^1} = \sum_i \hat{\mathcal{OC}}_{d,\tau_i} \tag{5.77}$$

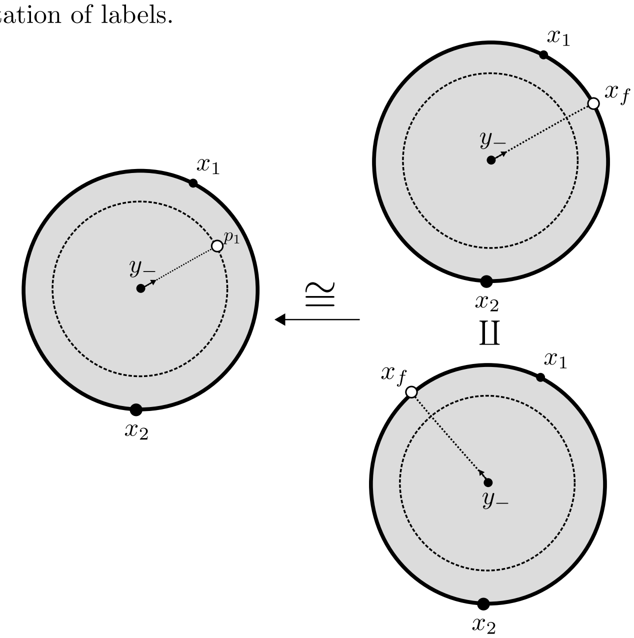
Proof. For each d, there is an embedding of abstract moduli spaces

$$\coprod_i \hat{\mathcal{R}}^1_{d,\tau_i} \xrightarrow{\coprod_i \pi_f^i} \coprod_i \mathcal{R}_d^{S^1_{i,i+1}} \hookrightarrow \mathcal{R}_d^{S^1}; \tag{5.78}$$

see Figure 3.

By construction, this map is compatible with Floer data (this uses the fact that the Floer data on $\mathcal{R}^{S^1_{i,i+1}}$ agrees with the data on $\hat{\mathcal{R}}^1_d$ via the reshuffling map κ^{-i} by (5.58)), and covers all but

FIGURE 3. The diffeomorphism between $\hat{\mathcal{R}}^1_{2,\tau_0} \cup \hat{\mathcal{R}}^1_{2,\tau_1}$ and the open dense part of $\mathcal{R}_2^{S^1}$ given by $\mathcal{R}_2^{S^1_{0,1}} \cup \mathcal{R}_2^{S^1_{1,2}}$. The former spaces can in turn be compared to $\hat{\mathcal{R}}^1_2$ via cyclic permutation of labels.



a codimension 1 locus in the target. Since after perturbation zero-dimensional solutions to Floer's equation can be chosen to come from the complement of any codimension 1 locus in the source abstract moduli space, we conclude that the two operations in the Proposition, which arise from either side of (5.78), are identical up to sign. To fix the signs, note that (5.78) is in fact an oriented embedding, and all the sign twists defining the operations $\hat{\mathcal{OC}}_{d,\tau_i}$ are chosen to be compatible with the sign twist in the operation $\mathcal{OC}^{S^1}$. $\square$

Next, because the Floer data used in the constructions are identical, $\hat{\mathcal{OC}}_{d,\tau_i}(x_d \otimes \cdots \otimes x_1)$ agrees with $\hat{\mathcal{OC}}(x_i \otimes \cdots \otimes x_1 \otimes x_d \otimes \cdots \otimes x_{i+1})$ up to a sign difference coming from orientations of abstract moduli spaces, cyclically reordering inputs, and sign twists. The following proposition computes the sign difference, and hence completes the proof of Proposition 7:

LEMMA 19. *As a signed operation,*

$$\hat{\mathcal{OC}}_{d,\tau_i}(x_d \otimes \cdots \otimes x_1) = \hat{\mathcal{OC}}^d(s^{nu}(t^i(x_d \otimes \cdots x_1))) \tag{5.79}$$

where s^{nu} is the operation (3.24) *arising from changing a check term to a hat term with a sign twist.*

PROOF. It is evident that $\hat{\mathcal{OC}}_{d,\tau_i}$ agrees with $\hat{\mathcal{OC}}_d \circ s^{nu} \circ t^i$ up to sign, as the Floer data used in the two constructions are identical. By an inductive argument it suffices to verify the following equalities of signed operations:

$$\hat{\mathcal{OC}}_{d,\tau_0} = \hat{\mathcal{OC}}_d \circ s^{nu}, \tag{5.80}$$

$$\hat{\mathcal{OC}}_{d,\tau_1} = \hat{\mathcal{OC}}_{d,\tau_0} \circ t; \tag{5.81}$$

the remaining sign changes are entirely incremental. For the equality (5.80), we simply note that the signs appearing in the operations $\hat{\mathcal{OC}}_{d,\tau_0}([x_d], \ldots, [x_1])$ and $\hat{\mathcal{OC}}_d([x_d], \ldots, [x_1])$ differ in the following fashions:

- The abstract orientations on the moduli space of domains agree, as per Remark 40.

- The difference in sign twists is given by $\clubsuit_d - \hat{\star}_d = \sum_{i=1}^d |x_i| + |x_d| + d - 1 = (\sum_{i=1}^d ||x_i||) + 1 + |x_d| = \maltese_1^d + ||x_d||$.

All together, the parity of difference in signs is $\maltese_1^d + ||x_d||$ which accounts for the sign in the algebraic operation s^{nu} (see (3.24)); this verifies (5.80).

Next, the sign difference between the two operations in the equality (5.81) is a sum of three contributions:

- The two orientations of abstract moduli spaces from (on the slice where z_f and z_{out} are fixed; see Remark 40) $-dz_1 \wedge \cdots \wedge dz_d$ to $dz_2 \wedge \cdots \wedge dz_d \wedge dz_1$ differ by a sign change of parity
$$d - 1.$$
- For a given collection of inputs, the change in *sign twisting data* from $\clubsuit_d = \sum_{i=1}^d (i+1) \cdot |x_i| + |x_d| + d - 1$ to $\sum_{i=1}^{d-1}(i+1)|x_{i+1}| + (d+1)|x_1| + |x_1| + d - 1 = \sum_{i=2}^d i|x_i| + d|x_1| + d - 1$ ($\clubsuit_d$ for the sequence $(x_2, \dots, x_d, x_1)$) induces a sign change of parity

$$\sum_{i=2}^d |x_i| + |x_d| + d|x_1| = \sum_{i=1}^d |x_i| + |x_d| + (d-1)|x_1| = \sum_{i=1}^d ||x_i|| + (d-1)||x_1|| + ||x_d|| = \maltese_1^d + (d-1)||x_1|| + ||x_d||.$$

- Finally, the re-ordering of determinant lines of the inputs induces a sign change of parity

$$|x_1| \cdot (\sum_{i=2}^d |x_i|) = ||x_1|| \cdot (\sum_{i=2}^d ||x_i||) + \sum_{i=2}^d ||x_i|| + (d-1)||x_1|| + (d-1) = ||x_1||\maltese_2^d + \maltese_1^d + d||x_1|| + (d-1)$$

The cumulative sign parity is congruent mod 2 to
$$||x_1||\maltese_2^d + ||x_1|| + ||x_d||,$$
which is precisely the sign appearing in t (see (3.11)). This verifies (5.81).

□

5.4. Compatibility of homology-level BV operators. Before diving into the statement of chain-level equivariance, we prove a homology-level statement. The below Theorem is insufficient for studying, say, equivariant homology groups, but may be of independent interest.

Theorem 5. *The homology level open-closed map $[\mathcal{OC}^{nu}]$ intertwines the Hochchild and symplectic cohomology BV operators, that is*

$$[\mathcal{OC}^{nu}] \circ [B^{nu}] = [\delta_1] \circ [\mathcal{OC}^{nu}]. \tag{5.82}$$

Theorem 5 is an immediate consequence of the following chain-level statement:

Proposition 8. *The following diagram homotopy commutes:*

$$\begin{tikzcd} \mathrm{CH}_{*-n}(\mathcal{F},\mathcal{F}) \arrow[r, hook, "\iota", "\sim"'] \arrow[d, "\check{\mathcal{OC}}"] & \mathrm{CH}^{nu}_{*-n}(\mathcal{F},\mathcal{F}) \arrow[r, "B^{nu}"] & \mathrm{CH}^{nu}_{*-n-1}(\mathcal{F},\mathcal{F}) \arrow[d, "\mathcal{OC}^{nu}"] \\ CF^*(M) \arrow[rr, "\delta_1"] & & CF^{*-1}(M). \end{tikzcd} \tag{5.83}$$

where ι is the inclusion onto the left factor, which is a quasi-isomorphism by Lemma 3. More precisely, there exists an operation $\check{\mathcal{OC}}^1 : \mathrm{CH}_{*-n}(\mathcal{F},\mathcal{F}) \to CF^{*-2}(M)$ satisfying

$$(-1)^{n+1} d\check{\mathcal{OC}}^1 + \check{\mathcal{OC}}^1 b = \hat{\mathcal{OC}} B^{nu}\iota - (-1)^n \delta_1 \check{\mathcal{OC}}. \tag{5.84}$$

Proof of Theorem 5. Proposition 8 immediately implies that $[\delta_1] \circ [\check{\mathcal{OC}}] = [\mathcal{OC}^{nu}] \circ [B^{nu}] \circ [\iota]$ where $\iota : \mathrm{CH}_{*-n}(\mathcal{F},\mathcal{F}) \to \mathrm{CH}^{nu}_{*-n}(\mathcal{F},\mathcal{F})$ is the inclusion of chain complexes. But by Lemma 3, $[\iota]$ is an isomorphism and by Corollary 6 $[\check{\mathcal{OC}}] = [\mathcal{OC}^{nu}]$. □

To define $\check{\mathcal{OC}}^1$, consider

$$ {}_1\check{\mathcal{R}}^1_{d+1}, \tag{5.85}$$

the moduli space of discs with with $d+1$ positive boundary marked points $z_0, \ldots, z_d$ labeled in counterclockwise order, 1 interior negative puncture z_{out} equipped with an asymptotic marker, and 1 additional interior marked points p_1 (without an asymptotic marker), marked as *auxiliary.* Also, choosing a representative of an element this moduli space which fixes z_0 at 1 and z_{out} at 0 on the unit disc, p_1 should lie *within a circle of radius* $\frac{1}{2}$:

$$0 < |p_1| < \frac{1}{2}. \tag{5.86}$$

Using the above representative, one can talk about the *angle*, or *argument* of p_1

$$\theta_i := \arg(p_i). \tag{5.87}$$

We require that with respect to the above representative,

$$\text{the asymptotic marker on } z_{out} \text{ points in the direction } \theta_1. \tag{5.88}$$

For every representative $S \in {}_1\check{\mathcal{R}}^1_{d+1}$,

$$\begin{gathered}\text{fix a negative cylindrical end around } z_{out} \text{ not containing } p_1\text{, compatible with the}\\ \text{direction of the asymptotic marker, or } \textit{equivalently compatible with the angle } \theta_1.\end{gathered} \tag{5.89}$$

We orient (5.85) as follows: pick, on a slice of the automorphism action which fixes the position of z_d at 1 and z_{out} at 0, the volume form

$$-r_1 dz_1 \wedge dz_2 \wedge \cdots \wedge dz_{d-1} \wedge dr_1 \wedge d\theta_1 \tag{5.90}$$

The compactification of (5.85) is a real blow-up of the ordinary Deligne-Mumford compactification, in the sense of [**KSV**] (see [**SS**] for a first discussion in the context of Floer theory).

The result of this discussion is that the codimension 1 boundary of the compactified check moduli space ${}_1\overline{\check{\mathcal{R}}}^1_{d+1}$ is covered by the images of the natural inclusions of the following strata:

$$\overline{\mathcal{R}^s} \times {}_1\overline{\check{\mathcal{R}}}^1_{d-s+2} \tag{5.91}$$

$$\overline{\check{\mathcal{R}}}^1_d \times \overline{\mathcal{M}}_1 \tag{5.92}$$

$$\overline{\check{\mathcal{R}}}^{S^1}_{d+1} \tag{5.93}$$

The stratum (5.93) describes the locus which $|p_1| = \frac{1}{2}$, which is exactly the locus we defined to be the auxiliary operation $\mathcal{R}^{S^1}_{d+1}$. The strata (5.91)-(5.92) have manifold with corners structure given by standard local gluing maps using fixed choices of strip-like ends near the boundary. For (5.91) this is standard, and for (5.92), the local gluing map uses the cylindrical ends (5.89) and (4.35) (in other words, one rotates the 1-pointed angle cylinder by an amount commensurate to the angle of the marked point z_d on the disk before gluing).

Definition 25. *A* universal and consistent Floer datum for the BV homotopy*, is an inductive choice, for every $d \geq 1$, of Floer data for every representative $S_0 \in {}_1\overline{\check{\mathcal{R}}}^1_{d+1}$, varying smoothly over moduli space, whose restriction to the boundary stratum is conformally equivalent to a product of Floer data coming from lower dimensional moduli spaces. Near the nodal boundary strata, with regards to gluing coordinates, this choice agrees to infinite order with the Floer data obtained by gluing.*

Proposition 9. *Universal and conformally consistent choices of Floer data for the BV homotopy exist.*

Fixing a universal and consistent choice of Floer data for the cyclic open-closed map, we obtain, for any d-tuple of Lagrangians $L_0, \ldots, L_{d-1}$, and asympotic conditions

$$\begin{aligned} \vec{x} &= (x_d, \ldots, x_1),\ x_i \in \chi(L_i, L_{i+1-\mathrm{mod} d}) \\ y_{out} &\in \mathcal{O} \end{aligned} \tag{5.94}$$

a compactified moduli space

$$ {}_1\overline{\check{\mathcal{R}}}^1_{d+1}(y_{out}, \vec{x}) \tag{5.95}$$

of maps into M with source an arbitrary element S of the moduli space (5.85), satisfying Floer's equation using the Floer datum chosen for the given S, and asymptotic and moving boundary conditions

$$\begin{cases} u(z) \in \psi^{\rho_S(z)} L_i & \text{if } z \in \partial S \text{ lies between } z_i \text{ and } z_{i+1 \text{ mod } d} \\ \lim_{s \to \pm\infty} u \circ \epsilon^k(s, \cdot) = x_k & \\ \lim_{s \to -\infty} u \circ \delta(s, \cdot) = y & \end{cases}. \tag{5.96}$$

Proposition 10. *For generic choices of Floer data, the Gromov-type compactifications* (5.95) *is a smooth compact manifold of dimension*

$$\dim({}_1\overline{\check{\mathcal{R}}}^1_{d+1}(y_{out}, \vec{x})) = \deg(y_{out}) - n + d + 1 - \sum_{i=0}^{d} \deg(x_i) \tag{5.97}$$

For rigid elements u in the moduli spaces (5.95), (which occurs for asymptotics $(y, \vec{x})$ satisfying (5.97) $= 0$), the orientations (5.90), and [**A**, Lemma C.4] induce isomorphisms of orientation lines

$$({}_1\check{\mathcal{R}}^1_d)_u : o_{x_d} \otimes \cdots \otimes o_{x_1} \to o_y \tag{5.98}$$

Summing the application of these isomorphisms over all u defines the $|o_{y_{out}}|_{\mathbf{k}}$ component of the operation $\check{\mathcal{OC}}^1$, up to a sign twist:

$$\check{\mathcal{OC}}^1([x_d], \ldots, [x_1]) := \sum_{\deg(y_{out}) = d-n-2k+1+\sum \deg(x_i)} \sum_{u \in {}_k\overline{\check{\mathcal{R}}}^1_d(y_{out}; \vec{x})} (-1)^{\check{\star}_d} ({}_k\check{\mathcal{R}}^1_d)_u([x_d], \ldots, [x_1]); \tag{5.99}$$

where the sign is given by

$$\check{\star}_d = \deg(x_d) + \sum_i i \cdot \deg(x_i). \tag{5.100}$$

A codimension 1 analysis of the moduli spaces (5.95) reveals:

Proposition 11. *The following equation is satisfied:*

$$(-1)^n \delta_1 \check{\mathcal{OC}} + (-1)^n d \check{\mathcal{OC}}^1 = \mathcal{OC}^{S^1} + \check{\mathcal{OC}}^1 b \tag{5.101}$$

Proof. In codimension 1, the boundary of (5.95) is covered by the following types of strata:

- spaces of maps with domain lying on the codimension 1 boundary of the moduli space, i.e., in (5.91)-(5.93)
- semi-stable breakings, namely those of the form

$$ {}_1\overline{\check{\mathcal{R}}}^1_d(y_1; \vec{x}) \times \overline{\mathcal{M}}(y_{out}; y_1) \tag{5.102}$$

$$ \overline{\mathcal{R}}^1(x_1; x) \times {}_1\overline{\check{\mathcal{R}}}^1_d(y_{out}; \tilde{\vec{x}}) \tag{5.103}$$

All together, this implies, up to signs, that

$$\delta_1 \check{\mathcal{OC}}^1 + d\mathcal{OC} = \mathcal{OC}^{S^1} + \check{\mathcal{OC}}^1 b. \tag{5.104}$$

(5.104) is of course a shorthand for saying, for a tuple of d cyclically composable morphisms $x_d, \ldots, x_1$, that

(5.105)

$$\begin{aligned}\sum_{i=0}^{1} \delta_i \check{\mathcal{OC}}_d^{k-i}(x_d, \ldots, x_1) = \mathcal{OC}_d^{S^1}(x_d, \ldots, x_1) + \sum_{i=1}^{k-1} \check{\mathcal{OC}}_d^{k,i,i+1}(x_d, \ldots, x_1) \\ + \sum_{i,s} (-1)^{\maltese_1^s} \check{\mathcal{OC}}^1_{d-i+1}(x_d, \ldots, x_{s+i+1}, \mu^i(x_{s+i}, \ldots, x_{s+1}), x_s, \ldots, x_1) \\ + \sum_{i,j} (-1)^{\sharp_j^i} \check{\mathcal{OC}}^1(\mu^{i+j+1}(x_i, \ldots, x_1, x_d, \ldots, x_{d-j}), x_{d-j-1}, \ldots, x_{i+1}).\end{aligned}$$

Thus, it suffices to verify that the signs coming from the codimension 1 boundary are exactly those appearing in (5.104). (in particular, that the terms in, for instance, $\check{\mathcal{OC}}^1 b$ appear with the right sign).

Let us recall broadly how the signs are computed. For any operator g defined above such as $\mathcal{OC}$, $\mathcal{OC}^{S^1}$, μ, d, δ_1 etc., we let g_{ut} denote the *untwisted* version of the same operator, e.g., the operator whose matrix coefficients come from the induced isomorphism on orientation lines, without any sign twists by the degree of the inputs. So for instance $\mu^d(x_d, \ldots, x_1) = (-1)^{\sum_{i=1}^d i \deg(x_i)} \mu^d_{\mathrm{ut}}(x_d, \ldots, x_1)$ and so on. The methods described in [**S5**, Prop. 12.3] and elaborated upon in [**A**, §C.3, Lemma 5.3] and [**G2**, §B], when applied to the boundary of the 1-dimensional component of the moduli space of maps, $\overline{\check{\mathcal{R}}}^1_{d+1}(y_{out}, \vec{x}))$, imply the following signed equality:

$$\begin{aligned}0 = d_{\mathrm{ut}} \check{\mathcal{OC}}^1_{\mathrm{ut}}(x_d, \ldots, x_1) + (\delta_1)_{\mathrm{ut}} \check{\mathcal{OC}}_{\mathrm{ut}}(x_d, \ldots, x_1) \\ - \mathcal{OC}^{S^1}_{\mathrm{ut}}(x_d, \ldots, x_1) + (-1)^{\mathfrak{f}_d} \check{\mathcal{OC}}^1 b(x_d, \ldots, x_1)\end{aligned} \tag{5.106}$$

where

$$\mathfrak{f}_d := \sum_i (i+1)\deg(x_i) + \deg(x_d) = \check{\star}_d + \maltese_d - d. \tag{5.107}$$

is an auxiliary sign.

To explain this equation (5.106), we note first that the signs appearing in all terms but the last are simply induced by the boundary orientation on the moduli space of domains. The sign appearing in the first term also follows from a standard boundary orientation analysis for Floer cylinders, which we omit (but see e.g., [**S5**, (12.19-12.20)] for a version close in spirit). The signs for the first two terms are also exactly as in Lemma 10. Finally, in the last term, the sign $(-1)^{\mathfrak{f}_d} \check{\mathcal{OC}}^1 b(x_d \otimes \cdots \otimes x_1)$ (compare [**S5**, (12.25)] [**G2**, (B.59)]) appears as a cumulative sum of

- the sign twists which turn the untwisted operations $\check{\mathcal{OC}}^1_{\mathrm{ut}}$ and μ^s_{ut} into the usual operations $\check{\mathcal{OC}}^1$ and μ^s;
- the Koszul sign appearing in the Hochschild differential b; and
- the boundary orientation sign appearing in the relevant (untwisted) term of $\check{\mathcal{OC}}^1 b$, for instance $\check{\mathcal{OC}}^1_{\mathrm{ut}}(x_d, \ldots, x_{n+m+1}, \mu^m_{ut}(x_{n+m}, \ldots, x_{n+1})x_n, \ldots, x_1)$, which itself is as a sum of two different contributions:
  - (a) the comparison between the boundary (of the chosen) orientation and the product (of the chosen orientation) on the moduli of *domains* and
  - (b) Koszul reordering signs, which measure the signed failure of the method of orienting the moduli of maps (in terms of orientations of the domain and orientation lines of inputs and outputs) to be compatible with passing to boundary strata.

  See [**S5**, (12d)] for more details in the case of the A_∞ structure, and [**A**, §C], [**G2**, §C] for the case of these computations for the open-closed map. We note in particular that the forgetful map $F_1 : {}_1\check{\mathcal{R}}^1_d \to \check{\mathcal{R}}^1_d$, which forgets the point p_1 (and changes the direction of the asymptotic marker to point at z_d) has *complex oriented fibers* (in which just the marked point p_1 varies). So the boundary analysis of these "$\check{\mathcal{OC}}^1 \circ b$" strata appearing here is identical to the analysis strata appearing in [**A**] and [**G2**] for the "$\mathcal{OC} \circ b$" strata, which is why we have not repeated it here.

Multiplying all terms of (5.106) by $(-1)^{\check{\star}_d + \maltese_d - d + 1}$, and noting that, for instance, $\maltese_d - d + 1 + n - 2 = \deg(\check{\mathcal{OC}}^1(x_d \otimes \cdots \otimes x_1))$, so

(5.108)
$$\begin{aligned}(-1)^{\check{\star}_d + \maltese_d - d + 1}(\delta_1)_{ut}\check{\mathcal{OC}}^1_{\mathrm{ut}}(x_d, \ldots, x_1) &= (-1)^{\deg(\check{\mathcal{OC}}^1(x_d,\ldots,x_1))-n}(\delta_1)_{ut}(-1)^{\check{\star}_d}\check{\mathcal{OC}}^1_{\mathrm{ut}}(x_d, \ldots, x_1)\\ &= \delta_1 \check{\mathcal{OC}}^1(x_d, \ldots, x_1),\end{aligned}$$

(and similarly for the $d \circ \mathcal{OC}^1$ term), it follows that

$$\begin{aligned}0 &= (-1)^n \delta_1 \check{\mathcal{OC}}(x_d, \ldots, x_1) + (-1)^n d\check{\mathcal{OC}}^1(x_d, \ldots, x_1)\\ &\quad - \check{\mathcal{OC}}^1 b(x_d, \ldots, x_1) - (-1)^{\check{\star}_d + \maltese_d - d + 1}\mathcal{OC}^{S^1}_{\mathrm{ut}}(x_d, \ldots, x_1),\end{aligned} \tag{5.109}$$

but $\check{\star}_d + \maltese_d - d + 1 = \clubsuit_d$, and hence the last term above is $-\mathcal{OC}^{S^1}(x_d, \ldots, x_1)$ as desired. $\square$

Proof of Proposition 8. The "sector decomposition" performed in Proposition 7 which compares $\mathcal{OC}^{S^1}$ to $\hat{\mathcal{OC}} \circ B^{nu} \circ \iota$, along with Proposition 10, immediately implies the result. $\square$

5.5. Higher cyclic chain homotopies and the main result. We now turn to the definition of the (closed) morphism of S^1-complexes, and the proof of Theorem 1 and Corollary 1. The required data takes the form

$$\widetilde{\mathcal{OC}} = \bigoplus_{k \geq 0} \overline{\mathbf{k}[\Lambda]/\Lambda^2}^{\otimes k} \otimes \mathrm{CH}^{nu}_*(\mathcal{F}, \mathcal{F}) \to CF^*(M)[n] \tag{5.110}$$

which is equivalent, as recalled in §2.1 to defining the collection of maps $\widetilde{\mathcal{OC}} = \{\mathcal{OC}^k\}_{k \geq 0}$, or u-linearly (see §2.3) $\widetilde{\mathcal{OC}} = \sum_{k=0}^{\infty} \mathcal{OC}^k u^k$, where

$$\mathcal{OC}^k = (\check{\mathcal{OC}}^k + \hat{\mathcal{OC}}^k) := \widetilde{\mathcal{OC}}^{k|1}(\Lambda, \ldots, \Lambda, -) : \mathrm{CH}^{nu}_*(\mathcal{F}, \mathcal{F}) \to CF^{*+n-2k}(M). \tag{5.111}$$

(recall from §2.1 that $\mathbf{k}[\Lambda]/\Lambda^2$ is our small model for $C_{-*}(S^1)$ and S^1-complexes are by definition strictly unital A_∞ modules over $\mathbf{k}[\Lambda]/\Lambda^2$). By definition, the case $k = 0$ is already covered:

$$\begin{aligned}\check{\mathcal{OC}}^0 &:= \check{\mathcal{OC}}\\ \hat{\mathcal{OC}}^0 &:= \hat{\mathcal{OC}}\end{aligned} \tag{5.112}$$

To handle the general case ($k \geq 0$), we will associate operations to, for each d, compactifications of three moduli spaces of domains, in the following order:

$${}_k\check{\mathcal{R}}^1_{d+1} \tag{5.113}$$

$${}_k\mathcal{R}^{S^1}_d \tag{5.114}$$

$${}_k\hat{\mathcal{R}}^1_d; \tag{5.115}$$

The moduli space (5.114) will induce an auxiliary operation useful for the proof, whereas (5.113) and (5.115) will lead to the desired operations. For $k = 0$, these moduli spaces are simply $\check{\mathcal{R}}^1_{d+1}$, $\mathcal{R}^{S^1}_d$, and $\hat{\mathcal{R}}^1_d$ as defined earlier. Inductively, we will construct, and study operations from (5.113) and (5.114) simultaneously, and then finally construct (5.115). Using these moduli spaces, we will construct the maps $\check{\mathcal{OC}}^k$, $\hat{\mathcal{OC}}^k$ and an auxiliary operation $\mathcal{OC}^{S^1,k}$ (which we compare to $\hat{\mathcal{OC}}^{k-1} \circ B^{nu}$ in Proposition 16 below), and then prove that:

Proposition 12. *The following equations hold, for each $k \geq 0$:*

$$(-1)^n \sum_{i\geq 0}^{k} \delta_i \check{\mathcal{OC}}^{k-i} = \hat{\mathcal{OC}}^{k-1} B^{nu} + \check{\mathcal{OC}}^{k} b \tag{5.116}$$

$$(-1)^n \sum_{i\geq 0}^{k} \delta_i \hat{\mathcal{OC}}^{k-i} = \hat{\mathcal{OC}}^{k} b' + \check{\mathcal{OC}}^{k}(1-t). \tag{5.117}$$

All at once, denoting by $\mathcal{OC}^k = (\check{\mathcal{OC}}^k + \hat{\mathcal{OC}}^k)$ and $\widetilde{\mathcal{OC}} = \sum_{i=0}^{\infty} \mathcal{OC}^i u^i$, $\delta_{eq} = \sum_{j=0}^{\infty} \delta_j^{CF} u^j$, and $b_{eq} = b^{nu} + uB^{nu}$ as in §2.3, we have that

$$(-1)^n \delta_{eq} \circ \widetilde{\mathcal{OC}} = \widetilde{\mathcal{OC}} \circ b_{eq}. \tag{5.118}$$

This will also directly imply our main Theorems, as we will spell out at the bottom of this subsection.

The space (5.113) is the moduli space of discs with with $d+1$ positive boundary marked points $z_0, \ldots, z_d$ labeled in counterclockwise order, 1 interior negative puncture z_{out} equipped with an asymptotic marker, and k additional interior marked points $p_1, \ldots, p_k$ (without an asymptotic marker), marked as *auxiliary*. Also, choosing a representative of an element this moduli space which fixes z_0 at 1 and z_{out} at 0 on the unit disc, the p_i should be *strictly radially ordered*; that is,

$$0 < |p_1| < \cdots < |p_k| < \frac{1}{2}. \tag{5.119}$$

Using the above representative, one can talk about the *angle*, or *argument* of each auxiliary interior marked point,

$$\theta_i := \arg(p_i). \tag{5.120}$$

We require that with respect to the above representative,

(5.121) the asymptotic marker on z_{out} points in the direction θ_1 (or towards z_0 if $k = 0$).

(equivalently one could define $\theta_{k+1} = 0$, so that θ_1 is always defined). For every representative $S \in {}_k\check{\mathcal{R}}^1_{d+1}$,

(5.122) fix a negative cylindrical end around z_{out} not containing any p_i, compatible with the direction of the asymptotic marker, or *equivalently compatible with the angle* θ_1.

The second moduli space (5.114) is the moduli space of discs with with d positive boundary marked points $z_1, \ldots, z_d$ labeled in counterclockwise order, 1 interior negative puncture z_{out} equipped with an asymptotic marker, and $k+1$ additional interior marked points $p_1, \ldots, p_k, p_{k+1}$ (without an asymptotic marker), marked as *auxiliary*. Choosing a representative of an element this moduli space which fixes z_0 at 1 and z_{out} at 0 on the unit disc, the p_i should be *strictly radially ordered* and p_{k+1} should lie on the circle of radius $\frac{1}{2}$; that is,

$$0 < |p_1| < \cdots < |p_k| < |p_{k+1}| = \frac{1}{2}. \tag{5.123}$$

This asymptotic marker for representative satisfies condition (5.121). Abstractly we have that ${}_k\mathcal{R}^{S^1}_d \cong S^1 \times {}_k\check{\mathcal{R}}^1_{d+1}$, where the S^1 parameter is given by the position of p_{k+1}.

The compactification of (5.113) is a real blow-up of the ordinary Deligne-Mumford compactification, in the sense of [**KSV**] (see [**SS**] for a first discussion in the context of Floer theory).

The result of this discussion is that the codimension 1 boundary of the compactified check moduli space ${}_k\overline{\check{\mathcal{R}}}^1_{d+1}$ is covered by the images of the natural inclusions of the following strata:

$$\overline{\mathcal{R}^s} \times {}_k\overline{\check{\mathcal{R}}}^1_{d-s+2} \tag{5.124}$$

$$ {}_s\overline{\check{\mathcal{R}}}^1_d \times \overline{\mathcal{M}}_{k-s} \tag{5.125}$$

$$ {}_{k-1}\overline{\check{\mathcal{R}}}^{S^1}_{d+1} \tag{5.126}$$

$$ {}_k^{i,i+1}\overline{\check{\mathcal{R}}}^1_{d+1} \tag{5.127}$$

The strata (5.126)-(5.127), in which $|p_k| = \frac{1}{2}$ and $|p_i| = |p_{i+1}|$ respectively, describe the boundary loci of the ordering condition (5.119) and hence come equipped with a natural manifold with corners structure. The strata (5.124)-(5.125) have manifold with corners structure given by standard local gluing maps using fixed choices of strip-like ends near the boundary. For (5.124) this is standard, and for (5.125), the local gluing map uses the cylindrical ends (5.122) and (4.35) (in other words, one rotates the r-pointed angle cylinder by an amount commensurate to the angle of the first marked point p_{k-s+1} on the disk before gluing).

Associated to the stratum (5.127) where p_i and p_{i+1} have coincident magnitudes, there is a forgetful map

$$\check{\pi}_i : {}_k^{i,i+1}\overline{\check{\mathcal{R}}}^1_{d+1} \to {}_{k-1}\overline{\check{\mathcal{R}}}^1_{d+1} \tag{5.128}$$

which simply forgets the point p_{i+1}. Since the norm of p_{i+1} and p_i agree on this locus, this amounts to forgetting the argument of p_{i+1} (in particular, the fibers of $\check{\pi}_i$ are one-dimensional).

The S^1-moduli space (5.114), is abstractly $S^1 \times {}_k\check{\mathcal{R}}^1_{d+1}$, and similarly we model its compactification is abstractly by $S^1 \times {}_k\overline{\check{\mathcal{R}}}^1_{d+1}$. However, it is preferable to give an explicit description of the boundary strata, which is covered in codimension 1 by the following strata:

$$\overline{\mathcal{R}^s} \times {}_k\overline{\mathcal{R}}^{S^1}_{d-s+2} \tag{5.129}$$

$$ {}_s\overline{\mathcal{R}}^{S^1}_d \times \overline{\mathcal{M}}_{k-s} \tag{5.130}$$

$$ {}_k^{k,k+1}\overline{\mathcal{R}}^{S^1}_{d+1} \tag{5.131}$$

$$ {}_k^{i,i+1}\overline{\mathcal{R}}^{S^1}_{d+1}. \tag{5.132}$$

Here, (5.129) and (5.130) are just versions of the degenerations (5.124) and (5.125), in which a collection of boundary points bubbles off, or a collection of auxiliary points convergest to z_{out} and bubbles off (the fact that the latter occurs in codimension 1 is part of the "real blow-up phenomenon" already discussed). The stratum (5.132), is the locus where $|p_i| = |p_{i+1}|$, for $i < k$, and the stratum (5.131), is the locus where $|p_k| = \frac{1}{2} = |p_{k+1}|$.

As in (5.128), on strata (5.131)-(5.132) where p_i and p_{i+1} have coincident magnitudes, define the map

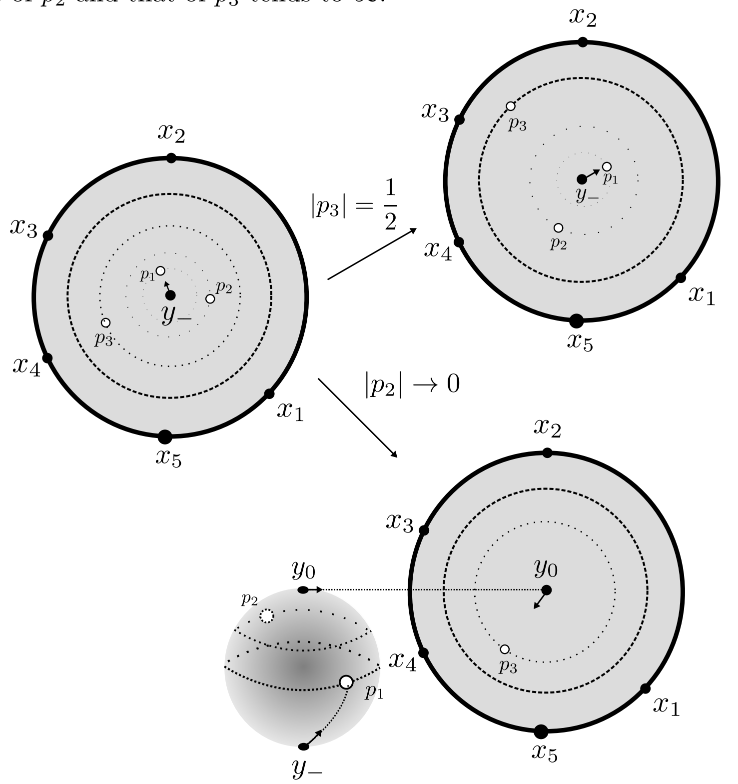
$$\pi_i^{S^1} : {}_k^{i,i+1}\overline{\mathcal{R}}^{S^1}_{d+1} \to {}_{k-1}\overline{\mathcal{R}}^{S^1}_{d+1} \tag{5.133}$$

to be the one forgetting the point p_{i+1} (so as before, this map has one-dimensional fibers), including the case $i = k$ which is (5.131).

For an element $S \in {}_k\overline{\mathcal{R}}^{S^1}_d$, we say that p_{k+1} *points at a boundary point* z_i if, for any unit disc representative of S with z_{out} at the origin, the ray from z_{out} to p_{k+1} intersects z_i. The locus where p_{k+1} points at z_i is denoted

$$ {}_k\overline{\mathcal{R}}^{S^1_i}_{d+1}, \tag{5.134}$$

FIGURE 4. A representative of an element of the moduli space ${}_3\check{\mathcal{R}}^1_5$, along with two of the most significant types of degenerations: in the first, the final auxiliary point $|p_3|$ reaches the circle of radius $\frac{1}{2}$, and one obtains an element of ${}_2\check{\mathcal{R}}_5^{S^1}$. In the second, one of the auxiliary points, $|p_2|$ tends to zero, forcing p_1 and p_2 into splitting off a copy of $\mathcal{M}_2$. In terms of the cylindrical coordinates near y_-, the distance between the height of p_2 and that of p_3 tends to ∞.



Similarly, we say that p_{k+1} *points between* z_i *and* z_{i+1} (modulo d, so including the case z_d *and* z_1) if for such a representative, the ray from z_{out} to p_{k+1} intersects the portion of ∂S between z_i and z_{i+1}. The locus where p_{k+1} points between z_i and z_{i+1} is denoted

$$ {}_k\overline{\mathcal{R}}_{d+1}^{S^1_{i,i+1}}. \tag{5.135}$$

As before (5.56), there is a free $\mathbb{Z}_d$ action

$$\kappa : {}_k\overline{(\mathcal{R}_d^1)^{S^1}} \to {}_k\overline{(\mathcal{R}_d^1)^{S^1}} \tag{5.136}$$

which cyclically permutes the labels of the boundary marked points; as before, κ changes the label z_i to z_{i+1} for $i < d$, and z_d to z_1.

Finally, we come to the third moduli space (5.115), the moduli space of discs with with $d+1$ positive boundary marked points $z_f, z_1, \ldots, z_d$ labeled in counterclockwise order, 1 interior negative puncture z_{out} equipped with an asymptotic marker, and k additional interior marked points $p_1, \ldots, p_k$ (without an asymptotic marker), marked as *auxiliary*, staisfying a *strict radial ordering* condition as before: for any representative element with z_f fixed at 1 and z_{out} at 0, we require (5.119) to hold, as well as condition (5.121). The boundary marked point z_f is also marked as auxiliary, but abstractly, we see that ${}_k\hat{\mathcal{R}}^1_d \cong {}_k\check{\mathcal{R}}^1_{d+1}$.

In codimension 1, the compactification ${}_k\overline{\hat{\mathcal{R}}}^1_d$ has boundary covered by inclusions of the following strata:

$$\overline{\mathcal{R}^s} \times {}_k\overline{\hat{\mathcal{R}}}^1_{d-s+2} \tag{5.137}$$

$$\overline{\mathcal{R}}^{m,f_k} \times_{d-m+1} {}_k\overline{\check{\mathcal{R}}}^1_{d-m+1} \quad 1 \leq k \leq m \tag{5.138}$$

$$ {}_s\overline{\hat{\mathcal{R}}}^1_d \times \overline{\mathcal{M}}_{k-s} \tag{5.139}$$

$$ {}_{k-1}\overline{\hat{\mathcal{R}}}^{S^1}_d \tag{5.140}$$

$$ {}^{i,i+1}_{k}\overline{\hat{\mathcal{R}}}^1_d \tag{5.141}$$

Once more, on strata (5.141) where p_i and p_{i+1} have coincident magnitudes, define the map

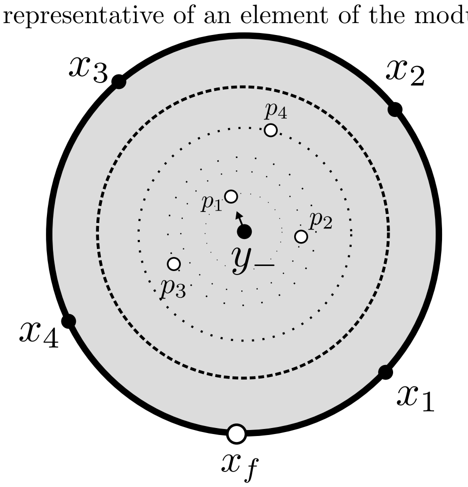
$$\hat{\pi}_i : {}^{i,i+1}_{k}\overline{\hat{\mathcal{R}}}^1_d \to {}_{k-1}\overline{\hat{\mathcal{R}}}^1_d. \tag{5.142}$$

to be the one forgetting the point p_{i+1} (so again, this map has one-dimensional fibers). On the stratum (5.140), which is the locus where $|p_k| = \frac{1}{2}$, there is also a map of interest

$$\hat{\pi}_{boundary} : {}_{k-1}\overline{\hat{\mathcal{R}}}^{S^1}_d \to {}_{k-1}\overline{\mathcal{R}}^{S^1}_d \tag{5.143}$$

which forgets the position of the auxiliary boundary point z_f.

FIGURE 5. A representative of an element of the moduli space ${}_4\hat{\mathcal{R}}^1_4$.



Denote by ${}_k\mathcal{R}^{1,free}_d := {}_k\mathcal{R}^{S^1_{d,1}}_d$ to be the sector of the moduli space ${}_k\mathcal{R}^{S^1}_d$ where p_{k+1} points between z_d and z_1. The *auxiliary-rescaling map*

$$\pi_f : {}_k\hat{\mathcal{R}}^1_d \to {}_k\mathcal{R}^{1,free}_d, \tag{5.144}$$

(our replacement of the "forgetful map") can be described as follows: given a representative S in ${}_k\hat{\mathcal{R}}^1_d$ with z_{out} fixed at the origin, there is a unique point p with $|p| = \frac{1}{2}$ between z_{out} and z_f. $\pi_f(S)$ is the element of ${}_k\mathcal{R}^{S^1}_d$ with p_{k+1} equal to this point p and with z_f deleted. Of course, z_f is not actually forgotten, because it is determined by the position of p_{k+1}. In particular (5.144) is a diffeomorphism.

We orient the moduli spaces (5.113)-(5.115) as follows: picking, on a slice of the automorphism action which fixes the position of z_d at 1 and z_{out} at 0, the volume forms

$$-r_1 \cdots r_k dz_1 \wedge dz_2 \wedge \cdots \wedge dz_{d-1} \wedge dr_1 \wedge d\theta_1 \wedge \cdots \wedge dr_k \wedge d\theta_k \tag{5.145}$$

$$r_1 \cdots r_k dz_1 \wedge dz_2 \wedge \cdots \wedge dz_{d-1} \wedge d\theta_{k+1} \wedge dr_1 \wedge d\theta_1 \wedge \cdots \wedge dr_k \wedge d\theta_k \tag{5.146}$$

$$r_1 \cdots r_k dz_1 \wedge dz_2 \wedge \cdots \wedge dz_{d-1} \wedge dz_f \wedge dr_1 \wedge d\theta_1 \wedge \cdots \wedge dr_k \wedge d\theta_k. \tag{5.147}$$

Above, (r_i, θ_i) denote the polar coordinate positions of the point p_i (we could equivalently use Cartesian coordinates (x_i, y_i) and substitute $dx_i \wedge dy_i$ for every instance of $r_i dr_i \wedge d\theta_i$, but polar coordinates are straightforwardly compatible with the boundary stratum where $|p_k| = \frac{1}{2}$).

Definition 26. *A* Floer datum *on a stable disc S in ${}_k\overline{\check{\mathcal{R}}}^1_{d+1}$ or a stable disc S in ${}_k\overline{\mathcal{R}}^{S^1}_d$ is simply a Floer datum for S in the sense of Definition 19. A* Floer datum *on a stable disc $S \in {}_k\overline{\hat{\mathcal{R}}}^1_d$ is a Floer datum for $\overline{\pi}_f(S)$.*

Definition 27. *A* universal and consistent Floer datum for the cyclic open-closed map, *is an inductive choice, for every $k \geq 0$ and $d \geq 1$, of Floer data for every representative $S_0 \in {}_k\overline{\check{\mathcal{R}}}^1_{d+1}$, $S_1 \in {}_k\overline{\mathcal{R}}^{S^1}_d$, and $S_2 \in {}_k\overline{\hat{\mathcal{R}}}^1_d$, varying smoothly over these three moduli spaces, whose restriction to the boundary stratum is conformally equivalent to a product of Floer data coming from lower dimensional moduli spaces. Near the nodal boundary strata, with regards to gluing coordinates, this choice agrees to infinite order with the Floer data obtained by gluing. Moreover, this choice should satisfy the following additional requirements: For $S_0 \in {}_k\overline{\check{\mathcal{R}}}^1_{d+1}$,*

(5.148) *At a boundary stratum of the form* (5.127)*, the Floer datum for S_0 is conformally equivalent to the one pulled back from ${}_{k-1}\overline{\check{\mathcal{R}}}^1_{d+1}$ via the forgetful map $\check{\pi}_i$.*

For $S_1 \in {}_k\overline{\mathcal{R}}^{S^1}_d$,

(5.149) *On the codimension-1 loci ${}_k\overline{(\mathcal{R}^1_d)^{S^1}_i}$ where p_{k+1} points at z_i, the Floer datum should agree with the pullback by τ_i of the existing Floer datum for the open-closed map.*

(5.150) *The Floer datum should be κ-equivariant, where κ is the map* (5.136)*.*

(5.151) *At a boundary stratum of the form* (5.131) *or* (5.132)*, the Floer datum for S_1 is conformally equivalent to the one pulled back from ${}_{k-1}\overline{\mathcal{R}}^{S^1}_{d+1}$ via the forgetful map $\pi^{S^1}_i$.*

Finally, for $S_2 \in {}_k\overline{\hat{\mathcal{R}}}^1_d$,

(5.152) *The choice of Floer datum on strata containing $\mathcal{R}^{d,f_i}$ components should be constant along fibers of the forgetful map $\mathcal{R}^{d,f_i} \to \mathcal{R}^{d-1}$.*

(5.153) *The Floer datum on the main component $(S_2)_0$ of $\pi_f(S_2)$ should coincide with the Floer datum chosen on $(S_2)_0 \in {}_k\mathcal{R}^{1,free}_d \subset {}_k\mathcal{R}^{S^1}_d$.*

(5.154) *At a boundary stratum of the form* (5.140)*, the Floer datum on the main component of S_2 is conformally equivalent to the one pulled back from ${}_k\overline{\mathcal{R}}^{S^1}_d$ via the forgetful map $\hat{\pi}_{boundary}$.*

(5.155) *At a boundary stratum of the form* (5.141)*, the Floer datum for S_2 is conformally equivalent to the one pulled back from ${}_{k-1}\overline{\hat{\mathcal{R}}}^1_{d+1}$ via the forgetful map $\hat{\pi}_i$.*

Proposition 13. *Universal and conformally consistent choices of Floer data for the cyclic open-closed map exist.*

Proof. Since the choices of Floer data at each stage are contractible, this follows from the straightforward verification that, for a suitably chosen inductive order on strata, the conditions satisfied by the Floer data at various strata do not contradict each other. We use the following inductive order: first, assume we've chosen Floer data inductively for all A_∞ operations and the non-unital open-closed map, which give us choices for ${}_0\check{\overline{\mathcal{R}}}_d$ and ${}_0\hat{\overline{\mathcal{R}}}_d$ for all d. Next, we choose Floer data for ${}_0\overline{\mathcal{R}}_d^{S^1}$; more precisely we choose this data before that of ${}_0\hat{\overline{\mathcal{R}}}_d$ and use the conditions above to induce that of ${}_0\hat{\overline{\mathcal{R}}}_d$. Inductively, assuming we have made all choices at level $k-1$ ($k>0$), we first choose Floer data for ${}_k\check{\overline{\mathcal{R}}}_d$ for each d, then ${}_k\overline{\mathcal{R}}_d^{S^1}$ for each d, and finally ${}_k\hat{\overline{\mathcal{R}}}_d$. $\square$

Fixing a universal and consistent choice of Floer data for the cyclic open-closed map, we obtain, for any d-tuple of Lagrangians $L_0,\dots,L_{d-1}$, and asympotic conditions

$$\begin{aligned} &\vec{x} = (x_d,\dots,x_1),\ x_i \in \chi(L_i, L_{i+1-\mathrm{mod} d}) \\ &y_{out} \in \mathcal{O} \end{aligned} \tag{5.156}$$

compactified moduli spaces

$$ {}_k\check{\overline{\mathcal{R}}}^1_{d+1}(y_{out},\vec{x}) \tag{5.157}$$

$$ {}_k\overline{\mathcal{R}}_d^{S^1}(y_{out},\vec{x}) \tag{5.158}$$

$$ {}_k\hat{\overline{\mathcal{R}}}^1_{d}(y_{out},\vec{x}) \tag{5.159}$$

of maps into M with source an arbitrary element S of the moduli spaces (5.113), (5.114), and (5.115) respectively (strictly speaking, for (5.115) the source is $\pi_f(S)$), satisfying Floer's equation using the Floer datum chosen for the given S, and asymptotic and moving boundary conditions

$$\begin{cases} u(z) \in \psi^{\rho_S(z)} L_i & \text{if } z\in\partial S \text{ lies between } z_i \text{ and } z_{i+1 \bmod d} \\ \lim_{s\to\pm\infty} u\circ\epsilon^k(s,\cdot) = x_k & \\ \lim_{s\to-\infty} u\circ\delta(s,\cdot) = y & \end{cases}. \tag{5.160}$$

Proposition 14. *For generic choices of Floer data, the Gromov-type compactifications* (5.157) - (5.159) *are smooth compact manifolds of dimension*

$$\dim({}_k\check{\overline{\mathcal{R}}}^1_{d+1}(y_{out},\vec{x})) = \deg(y_{out}) - n + d - 1 - \sum_{i=0}^d \deg(x_i) + 2k; \tag{5.161}$$

$$\dim({}_k\overline{\mathcal{R}}_d^{S^1}(y_{out},\vec{x})) = \deg(y_{out}) - n + d - \sum_{i=0}^d \deg(x_i) + 2k; \tag{5.162}$$

$$\dim({}_k\hat{\overline{\mathcal{R}}}^1_{d}(y_{out},\vec{x})) = \deg(y_{out}) - n + d - \sum_{i=0}^d \deg(x_i) + 2k. \tag{5.163}$$

For rigid elements u in the moduli spaces (5.157) - (5.159), (which occurs for asymptotics $(y,\vec{x})$ satisfying (5.161) $=0$, (5.162) $=0$, or (5.163) $=0$ respectively), the orientations (5.145), (5.146), (5.147) and [**A**, Lemma C.4] induce isomorphisms of orientation lines

$$({}_k\check{\mathcal{R}}^1_{d+1})_u : o_{x_d}\otimes\cdots\otimes o_{x_1} \to o_y \tag{5.164}$$

$$({}_k\mathcal{R}_d^{S^1})_u : o_{x_d}\otimes\cdots\otimes o_{x_1} \to o_y \tag{5.165}$$

$$({}_k\hat{\mathcal{R}}^1_{d})_u : o_{x_d}\otimes\cdots\otimes o_{x_1} \to o_y. \tag{5.166}$$

Summing the application of these isomorphisms over all u defines the $|o_{y_{out}}|_{\mathbf{k}}$ component of three families of operations $\check{\mathcal{OC}}^k$, $\mathcal{OC}^{S^1,k}$, $\hat{\mathcal{OC}}^k$ up to a sign twist:

(5.167)
$$\check{\mathcal{OC}}^k([x_d],\ldots,[x_1]) := \sum_{\deg(y_{out})=d-n-2k+1+\sum\deg(x_i)} \sum_{u\in {}_k\overline{\check{\mathcal{R}}}^1_d(y_{out};\vec{x})} (-1)^{\check{\star}_d}({}_k\check{\mathcal{R}}^1_d)_u([x_d],\ldots,[x_1]);$$

(5.168)
$$\mathcal{OC}^{S^1,k}([x_d],\ldots,[x_1]) := \sum_{\deg(y_{out})=d-n-2k+\sum\deg(x_i)} \sum_{u\in {}_k\overline{\mathcal{R}}^{S^1}_d(y_{out};\vec{x})} (-1)^{\star^{S^1}_d}({}_k\mathcal{R}^{S^1}_d)_u([x_d],\ldots,[x_1]);$$

(5.169)
$$\hat{\mathcal{OC}}^k([x_d],\ldots,[x_1]) := \sum_{\deg(y_{out})=d-n-2k+\sum\deg(x_i)} \sum_{u\in {}_k\overline{\hat{\mathcal{R}}}^1_d(y_{out};\vec{x})} (-1)^{\hat{\star}_d}({}_k\hat{\mathcal{R}}^1_d)_u([x_d],\ldots,[x_1]).$$

where the signs are given by

$$\check{\star}_d = \deg(x_d) + \sum_i i\cdot\deg(x_i). \tag{5.170}$$

$$\star^{S^1}_d = \clubsuit_d = \sum_{i=1}^d (i+1)\cdot\deg(x_i) + \deg(x_d) + d - 1 = \check{\star}_d + \maltese_d - 1 \tag{5.171}$$

$$\hat{\star}_d = \sum_i i\cdot\deg(x_i). \tag{5.172}$$

A codimension 1 analysis of the moduli spaces (5.157) and (5.159) reveals:

Proposition 15. *The following equations hold for each $k\geq 0$:*

$$(-1)^n\sum_{i=0}^k \delta_i\check{\mathcal{OC}}^{k-i} = \mathcal{OC}^{S^1,k-1} + \check{\mathcal{OC}}^k b \tag{5.173}$$

$$(-1)^n\sum_{i=0}^k \delta_i\hat{\mathcal{OC}}^{k-i} = \hat{\mathcal{OC}}^k b' + \check{\mathcal{OC}}^k(1-t). \tag{5.174}$$

Proof. In codimension 1, the boundary of (5.157) is covered by the following types of strata:

- spaces of maps with domain lying on the codimension 1 boundary of the moduli space, i.e., in (5.124)-(5.127)
- semi-stable breakings, namely those of the form

$${}_k\overline{\check{\mathcal{R}}}^1_d(y_1;\vec{x}) \times \overline{\mathcal{M}}(y_{out};y_1) \tag{5.175}$$

$$\overline{\mathcal{R}}^1(x_1;x) \times {}_k\overline{\check{\mathcal{R}}}^1_d(y_{out};\tilde{\vec{x}}) \tag{5.176}$$

All together, this implies, up to sign, that

$$(-1)^n\sum_{i=0}^k \delta_i\check{\mathcal{OC}}^{k-i} = \mathcal{OC}^{S^1,k-1} + \check{\mathcal{OC}}^k b + \sum_{i=1}^{k-1}\check{\mathcal{OC}}^{k,i,i+1}, \tag{5.177}$$

where $\check{\mathcal{OC}}^{k,i,i+1}$ is an operation corresponding with some sign twist to (5.127). (5.177) is of course a shorthand for saying, for a tuple of d cyclically composable morphisms $x_d, \ldots, x_1$, that

(5.178)
$$\begin{aligned}\sum_{i=0}^{k} \delta_i \check{\mathcal{OC}}_d^{k-i}(x_d, \ldots, x_1) = \mathcal{OC}_d^{S^1, k-1}(x_d, \ldots, x_1) + \sum_{i=1}^{k-1} \check{\mathcal{OC}}_d^{k,i,i+1}(x_d, \ldots, x_1) \\ + \sum_{i,s} (-1)^{\maltese_1^s} \check{\mathcal{OC}}_{d-i+1}^{k}(x_d, \ldots, x_{s+i+1}, \mu^i(x_{s+i}, \ldots, x_{s+1}), x_s, \ldots, x_1) \\ + \sum_{i,j} (-1)^{\sharp_j^i} \check{\mathcal{OC}}^k(\mu^{i+j+1}(x_i, \ldots, x_1, x_d, \ldots, x_{d-j}), x_{d-j-1}, \ldots, x_{i+1}).\end{aligned}$$

We first note that in fact the operation $\check{\mathcal{OC}}^{k,i,i+1} = \sum_d \check{\mathcal{OC}}_d^{k,i,i+1}$ is zero, because by condition (5.148), the Floer datum chosen for elements S in (5.127) are constant along the one-dimensional fibers of $\check{\pi}_i$. Hence, elements of the moduli space with source in (5.127) are never rigid (see Lemma 10 for an analogous more detailed explanation).

Thus, it suffices to verify that the signs coming from the codimension 1 boundary is exactly that appearing in (5.177). We can safely ignore studying any signs for the vanishing operations such as $\hat{\mathcal{OC}}^{k,i,i+1}$. The remaining sign analysis is exactly as in Proposition 11; more precisely note that the forgetful map $\check{F}_k : {}_k\check{\mathcal{R}}_d^1 \to {}_1\check{\mathcal{R}}_d^1$ which forgets $p_1, \ldots, p_{k-1}$ has complex oriented fibers, and in particular (since the marked points p_i contribute complex domain orientations and do not introduce any new orientation lines) the sign computations sketched in Proposition 11 carry over for any strata whose domain is pulled back from a boundary stratum of ${}_1\check{\mathcal{R}}_d^1$ (in turn, as described in Proposition 11, the sign computations for ${}_1\check{\mathcal{R}}_d^1$ largely reduce to those for ${}_0\check{\mathcal{R}}_d^1$. This verifies (5.177).

Similarly, for the hat moduli space, an analysis of the boundary of 1-dimensional moduli spaces of maps tells us, up to sign verification:

$$(-1)^n \sum_{i=0}^{k} \delta_i \hat{\mathcal{OC}}^{k-i} = \hat{\mathcal{OC}}^k b' + \check{\mathcal{OC}}^k (1-t) + \hat{\mathcal{OC}}^{k,k,k+1} + \sum_{i=1}^{k-1} \hat{\mathcal{OC}}^{k,i,i+1}, \tag{5.179}$$

where $\hat{\mathcal{OC}}^{k,k,k+1}$ is an operation corresponding with some sign twist to (5.140) and $\hat{\mathcal{OC}}^{k,i,i+1}$ is an operation corresponding with some sign twist to (5.141). The conditions (5.154)-(5.155) similarly imply that $\hat{\mathcal{OC}}^{k,k,k+1}$ and $\hat{\mathcal{OC}}^{k,i,i+1}$ are zero, so it is not necessary to even establish what the signs for these terms are.

To verify signs for (5.179), we apply the principle discussed in the proof of Lemma 17, in which by treating the auxiliary boundary marked point z_f as possessing a "formal unit element asymptotic constraint e_+," therefore viewing $\hat{\mathcal{OC}}^k(x_d \otimes \cdots \otimes x_)$ formally as "$\hat{\mathcal{OC}}^k(e^+ \otimes x_d \otimes \cdots \otimes x_1)$" the signs for the equations (5.179) applied to strings of length d $(x_d \otimes \cdots \otimes x_1)$ follow follow from the sign computations for $\check{\mathcal{OC}}$ applied to strings of length $d+1$ $(e^+ \otimes x_d \cdots \otimes \cdots x_1)$ (we note that this analysis applies to the term $\hat{\mathcal{OC}}^{k,k,k+1}$ as well, which is the hat version of $\mathcal{OC}^{S^1,k}$; however, the former operation happens to be zero because extra symmetries imply the moduli space controlling this operation is never rigid).

□

Next, by decomposing the moduli space ${}_k\mathcal{R}_d^{S^1}$ into sectors, we can write the auxiliary operation $\mathcal{OC}^{S^1,j}$ in terms of $\hat{\mathcal{OC}}^j$ and the Connes' B operator:

Proposition 16. *As chain-level operations,*

$$\mathcal{OC}^{S^1,k} = \hat{\mathcal{OC}}^k \circ B^{nu}. \tag{5.180}$$

Proof. The proof directly emulates Proposition 7, and as such we will give fewer details. We begin by defining operations

$$\hat{\mathcal{OC}}^k_{d,\tau_i} \tag{5.181}$$

associated to various "sectors" of the $k+1$st marked point p_{k+1} of ${}_k\mathcal{R}^{S^1}_d$, for $i \in \mathbb{Z}/d\mathbb{Z}$. Once more, to gain better control of the geometry of these sectors in the compactification (when the sector size can shrink to zero), we pass to an alternate model for the compactification: define

$${}_k\mathcal{R}^1_{d,\tau_i} \tag{5.182}$$

to be the abstract moduli space of discs with $d+1$ boundary punctures, $z_1, \ldots, z_i, z_f, z_{i+1}, \ldots, z_d$ arranged in clockwise order, one interior negative puncture z_{out} with asymptotic marker, and k additional interior auxiliary marked points $p_1, \ldots, p_k$ which are *strictly ordered*; for a representative fixing z_0 at 1 and z_{out} at 0,

$$0 < |p_1| < \ldots < |p_k| < \frac{1}{2}. \tag{5.183}$$

Moreover, as before,

(5.184) the asymptotic marker on z_{out} points in the direction θ_1 (or towards z_f if $k=0$).

There is a bijection

$$\tau_i : {}_k\mathcal{R}^1_{d,\tau_i} \to {}_k\hat{\mathcal{R}}^1_d \tag{5.185}$$

given by cyclically permuting boundary labels, and in particular we also have an *auxiliary-rescaling map* as in (5.144)

$${}_k\mathcal{R}^1_{d,\tau_i} \to {}_k\mathcal{R}^{S^1_{i,i+1}}_d, \tag{5.186}$$

which, for a representative with $|z_{out}| = 0$, adds a point p_{k+1} on the line between z_{out} and z_f with $|p_{k+1}| = \frac{1}{2}$ and deletes z_f. We choose orientations on ${}_k\mathcal{R}^1_{d,\tau_i}$ to be compatible with (5.186); more concretely, for a slice fixing the positions of z_{out} and z_d, consider the top form

$$r_1 \cdots r_k dz_1 \wedge dz_2 \wedge \cdots \wedge dz_{d-1} \wedge dz_d \wedge dz_f \wedge dr_1 \wedge d\theta_1 \wedge \cdots \wedge dr_k \wedge d\theta_k \tag{5.187}$$

The compactification ${}_k\overline{\mathcal{R}}^1_{d,\tau_i}$ is inherited from the identification (5.185); the salient point is that we treat bubbled off boundary strata containing the point z_f as coming from $\mathcal{R}^{d,f_i}$, the moduli space of discs with ith marked point forgotten (where the ith marked point is z_f), constructed in Appendix A.2.

We choose as a Floer datum for ${}_k\overline{\mathcal{R}}^1_{d,\tau_i}$ the pulled back Floer datum from (5.185), it automatically then exists and is universal and consistent as desired. Moreover we have chosen orientations as in the case $k=0$ so that the auxiliary rescaling map (5.186) is an oriented diffeomorphism extending to a map between compactifications.

Thus, given a Lagrangian labeling $\{L_0, \ldots, L_{d-1}\}$ and compatible asymptotics $\{x_1, \ldots, x_d; y_{out}\}$ we obtain a moduli space of maps satisfying Floer's equation with the chosen boundary and asymptotics:

$${}_k\overline{\mathcal{R}}^1_{d,\tau_i}(y_{out}; \vec{x}) := {}_k\overline{\hat{\mathcal{R}}}^1_d(y_{out}; x_{i-1}, \ldots, x_1, x_d, \ldots, x_i) \tag{5.188}$$

a smooth manifold of dimension equal to the dimension of the right hand side, $\deg(y_{out}) - n + d - \sum_{j=1}^d \deg(x_j) + 2k$. The isomorphisms of orientation lines

$$({}_k\mathcal{R}^1_{d,\tau_i})_u : o_{x_d} \otimes \cdots \otimes o_{x_1} \to o_{y_{out}} \tag{5.189}$$

induced by elements u of the zero-dimensional components of (5.188) define the $|o_{y_{out}}|_{\mathbf{k}}$ component of the operation $\hat{\mathcal{OC}}^k_{d,\tau_i}$, up to the following sign twist:

(5.190)

$$\hat{\mathcal{OC}}^k_{d,\tau_i}([x_d],\ldots,[x_1]) := \sum_{\deg(y_{out})=n-d+\sum \deg(x_k)-2k} \sum_{u\in {}_k\overline{\hat{\mathcal{R}}}^1_{d,\tau_i}(y_{out};\vec{x})} (-1)^{\hat{\star}_d}({}_k\mathcal{R}^1_{d,\tau_i})_u([x_d],\ldots,[x_1]),$$

$$\star^{S^1}_d = \sum_{i=1}^d (i+1)\cdot\deg(x_i) + \deg(x_d) + d - 1.$$

Now, exactly as in Lemma 18, there is a chain-level equality of signed operations

$$\mathcal{OC}^{S^1,k}_d = \sum_{i=0}^{d-1} \hat{\mathcal{OC}}^k_{d,\tau_i}. \tag{5.191}$$

We recall the geometric statement underlying this; the point is by construction there is an oriented embedding

$$\coprod_i {}_k\mathcal{R}^1_{d,\tau_i} \stackrel{\coprod_i \pi^i_f}{\longrightarrow} \coprod_i {}_k\mathcal{R}^{S^1_{i,i+1}}_d \hookrightarrow {}_k\mathcal{R}^{S^1}_d, \tag{5.192}$$

compatible with Floer data, covering all but a codimension 1 locus in the target, and moreover all the sign twists defining the operations $\mathcal{OC}^k_{d,\tau_i}$ are chosen to be compatible with the sign twist in the operation $\mathcal{OC}^{S^1,k}$ (this uses the fact that the Floer data on ${}_k\mathcal{R}^{S^1_{i,i+1}}$ agrees with the data on ${}_k\hat{\mathcal{R}}^1_d$ via the cyclic permutation map κ^{-i} by (5.58)). After perturbation zero-dimensional solutions to Floer's equation can be chosen to come from the complement of any codimension 1 locus in the source abstract moduli space, implying the equality (5.191).

Finally, all that remains is a sign analysis whose conclusion is that

$$\hat{\mathcal{OC}}^k_{d,\tau_i} = \hat{\mathcal{OC}}^k_d \circ s^{nu} \circ t^i \tag{5.193}$$

where s^{nu} is the operation arising from changing a check term to a hat term with a sign twist (3.24). (some such statement is unsurprising, as the operations are constructed with identical Floer data but potentially different sign twists). The details of this sign comparison are exactly the same as in Proposition 19, including with signs, as when orienting the moduli of maps, the additional marked points $p_1,\ldots p_k$ only contribute complex orientations to the moduli spaces of domains (and no additional orientation line terms). □

Proof of Proposition 12. This is an immediate corollary of the previous two propositions. □

We now collect all of this information to finish the proof of our main result.

Proof of Theorem 1. The pre-morphism $\widetilde{\mathcal{OC}} \in \mathrm{Rhom}^n_{S^1}(\mathrm{CH}^{nu}_*(\mathcal{F},\mathcal{F}), CF^*(M))$, written u-linearly as $\sum_i \mathcal{OC}^k u^k$, where $\mathcal{OC}^k = \check{\mathcal{OC}}^k \oplus \hat{\mathcal{OC}}^k$ are as constructed above, satisfies $\partial\widetilde{\mathcal{OC}} = 0$ by Prop. 12, hence $\widetilde{\mathcal{OC}}$ is closed, or an S^1-complex homomorphism, also known as an A_∞ $C_{-*}(S^1)$ module homomorphism (see §2.1). Note that $[\mathcal{OC}^0] = [\mathcal{OC}^{nu}] = [\mathcal{OC}]$ where the first equality is by definition and the second is by Corollary 6, hence $\widetilde{\mathcal{OC}}$ is an enhancement of $\mathcal{OC}$ (as defined in §2.1). □

Proof of Theorem 1. This is an immediate consequence of Theorem 1 and the induced homotopy-invariance properties for equivariant homology groups discussed in §2, particularly Cor. 3 and Prop. 2. □

5.6. Variants of the cyclic open-closed map.

5.6.1. *Using singular (pseudo-)cycles instead of Morse cycles.* Let M be Liouville or compact and monotone (in which case $\bar{M} = M$ and $\partial\bar{M} = \emptyset$,[9] and let us consider the version of $\widetilde{\mathcal{OC}}$ with target the relative cohomology $H^*(\bar{M}, \partial\bar{M})$ as in §4.1.2. Instead of using a C^2 small Hamiltonian to define the Floer complex computing $H^{*+n}(M, \partial M)$ (which we only did for simultaneous compatibility with the symplectic cohomology case), we can pass to a geometric cycle model for the group, and then the map $\widetilde{\mathcal{OC}}$, which simplifies many of the constructions in the previous section (in the sense that the codimension 1 boundary strata of moduli spaces, and hence the equations satisfied by $\widetilde{\mathcal{OC}}$, are strictly a subset of the terms appearing above). As such, it will be sufficient to fix some notation for the relevant moduli spaces, and state the relevant simplified results.

We denote by

$$ {}_k\check{\mathcal{P}}^1_{d+1} \tag{5.194}$$

$$ {}_k\mathcal{P}^{S^1}_{d} \tag{5.195}$$

$$ {}_k\hat{\mathcal{P}}^1_{d}; \tag{5.196}$$

copies of the abstract moduli spaces (5.113)-(5.115) where the interior puncture z_{out} is filled in and replaced by a marked point $\bar{z}_{out}$, *without any asymptotic marker.* The compactifications of these moduli spaces are exactly as before, except that the auxiliary points $p_1, \ldots, p_k$ are now allowed to coincide with $\bar{z}_{out}$, without breaking off an angle-decorated cylinder / element of $\mathcal{M}_r$ (in the language of §4.3. In other words, the real blow-up of Deligne-Mumford compactifications at z_{out}, which is responsible for boundary strata containing $\mathcal{M}_r$ factors, *no longer occurs* (but all other degenerations do occur). Correspondingly the codimension-1 boundaries have all of the factors as before except for factors containing $\mathcal{M}_r$'s.

Inductively choose consistent Floer data as before on these moduli spaces of domains, satisfying all of the consistency conditions as before (except for any consistency conditions involving $\mathcal{M}_r$ moduli spaces, which no longer occur on the boundary). For a basis $\beta_1, \ldots, \beta_s$ of smooth (pseudo)-cycles in homology $H_*(M)$ whose Poincaré duals $[\beta_i^\vee]$ generate the cohomology $H^*(\bar{M}, \partial\bar{M})$, one obtains moduli spaces

$$ {}_k\check{\mathcal{P}}^1_{d+1}(\beta_i; \vec{x}) \tag{5.197}$$

$$ {}_k\mathcal{P}^{S^1}_{d}(\beta_i, \vec{x}) \tag{5.198}$$

$$ {}_k\hat{\mathcal{P}}^1_{d}(\beta_i, \vec{x}); \tag{5.199}$$

of moduli spaces of maps into M with source an arbitrary element of the relevant domain moduli space, satisfying Floer's equation as before, with Lagrangian boundary and asymptotics $\vec{x}$ as before, *with the additional point constraint that z_{out} lie on the cycle β_i.* As before, standard methods ensure that 0 and 1-dimensional moduli spaces are (for generic choices of perturbation data and/or β_i) transversely cut out manifolds of the "right" dimension and boundary, which is all that we need.

Then, define the coefficient of $[\beta_i^\vee] \in H^*(\bar{M}, \partial\bar{M})$ in $\check{\mathcal{OC}}^k(x_d \otimes \cdots \otimes x_1)$ to be given by signed counts (with the same sign twists as before) of the moduli spaces (5.197); similarly for $\hat{\mathcal{OC}}^k$ and $\mathcal{OC}^{S^1,k}$ using the moduli spaces (5.199) and (5.198). A simplification of the arguments already given (in which the δ_k operations no longer occur, but every other part of the argument carries through) implies that:

PROPOSITION 17. *The pre-morphism $\widetilde{\mathcal{OC}} = \sum_{i=0}^\infty \mathcal{OC}^k u^k \in \mathrm{Rhom}^n_{S^1}(\mathrm{CH}^{nu}_*(\mathcal{F}, \mathcal{F}), H^*(\bar{M}, \partial\bar{M}))$ satisfies*

$$ \widetilde{\mathcal{OC}} \circ b_{eq} = 0, \tag{5.200}$$

[9] Technically we should write $\mathrm{QH}^*(M)$ in this case, but additively $\mathrm{QH}^*(M) = H^*(M)$ and correspondingly no sphere bubbling occurs in the moduli spaces we define here, so there is no difference–yet.

where $b_{eq} = b^{nu} + uB^{nu}$. In other words, $\widetilde{\mathcal{OC}}$ is a homomorphism of S^1-complexes between $\mathrm{CH}_*^{nu}(\mathcal{F}, \mathcal{F})$ with its strict S^1 action and $H^*(\bar{M}, \partial\bar{M})$ with its trivial S^1 action.

5.6.2. *Compact Lagrangians in non-compact manifolds.* Now let us explicitly restrict to the case of M a Liouville manifold, and denote by $\mathcal{F} \subset \mathcal{W}$ the full-subcategory consisting of a finite collection of compact exact Lagrangian branes contained in the compact region $\overline{M}$. In this case, thinking of the map $\mathcal{OC}$ (and its cyclic analogue, $\widetilde{\mathcal{OC}}$) as a pairing $\mathrm{CH}_*^{nu}(\mathcal{F}, \mathcal{F}) \otimes H^*(M) \to \mathbf{k}[n]$, there is a non-trivial refinement

$$\mathcal{OC}_{cpct} : \mathrm{CH}_*(\mathcal{F}) \otimes SC^*(M) \to \mathbf{k}[-n]. \tag{5.201}$$

where $SC^*(M)$ is the *symplectic cohomology* chain complex.

Remark 41. The refinement (5.201) relies on extra flexibility in Floer theory for compact Lagrangians, first alluded to in this form in [**S6**], which allow us to define operations lacking outputs by Poincaré dually treating some inputs as outputs with “negative weight”, something not possible for the wrapped Fukaya category. Or, equivalently, the *sub-closed one-form* α_S used in Floer theoretic perturbations has complete freedom along boundary conditions corresponding to compact Lagrangians, whereas along non-compact boundary conditions, it is required to vanish in order to obtain desired compactness results. In particular, it is possible for α_S to be sub-closed even if the the sum of weights of outputs minus the sum of weights of inputs is not positive.

Remark 42. The idea that there should be a map $SC^*(M) \to \mathrm{CH}_*(\mathcal{F})^\vee$ is in some sense not new. Namely, categories $\mathcal{C}$ with a *weak proper Calabi-Yau structure* of dimension n (such as the Fukaya category of compact Lagrangians; see e.g., [**S5**, (12j)]) come equipped with isomorphisms between the dual of Hochschild chains and Hochschild co-chains $\mathrm{CH}_*(\mathcal{C})^\vee[-n] \simeq \mathrm{CH}^*(\mathcal{C})$, and the existence of a map $SH^*(M) \to \mathrm{HH}^*(\mathcal{F})$ was already observed in [**S2**].

The geometric moduli spaces used to prove our main result apply verbatim in this case (with the interior marked point changed to an input, and the ordering of the auxiliary marked points reversed). We will simply state the resulting structures:

Proposition 18. *Consider $\mathrm{CH}_*(\mathcal{F}, \mathcal{F}) \otimes SC^*(M)$ as an S^1-complex with its diagonal S^1 action (see Lemma 1 in §2.1), and $\mathbf{k} \in S^1{-}\mathrm{mod}$ with its trivial action. Then, there is a homomorphism of S^1-complexes*

$$\widetilde{\mathcal{OC}}_{cpct} \in \mathrm{Rhom}^n_{S^1}(\mathrm{CH}_*(\mathcal{F}, \mathcal{F}) \otimes SC^*(M), \mathbf{k})$$

e.g., $\widetilde{\mathcal{OC}}_{cpct}$ satisfies $\partial\widetilde{\mathcal{OC}}_{cpct} = 0$; in other words, for any $\sigma \otimes y \in \mathrm{CH}_*(\mathcal{F}, \mathcal{F}) \otimes SC^*(M)$ (or linear combination thereof), we have

$$\widetilde{\mathcal{OC}}_{cpct,eq} \circ ((-1)^{\deg(y)} b_{eq}(\sigma) \otimes y + \sigma \otimes \delta^{SC}_{eq}(y)) = 0.$$

To clarify the relevant moduli spaces used, we define

$$_k\check{\mathcal{R}}^1_{d+1,cpct} \tag{5.202}$$

$$_k\mathcal{R}^{S^1}_{d,cpct} \tag{5.203}$$

$$_k\hat{\mathcal{R}}^1_{d,cpct}; \tag{5.204}$$

copies of the abstract moduli spaces (5.113)-(5.115) where the interior puncture z_{out} is now a *positive* puncture (still equipped with an asymptotic marker); and all of the other inputs and auxiliary points are as before. The compactified moduli spaces have boundary strata agreeing with the boundary strata of the compactified (5.113)-(5.115), except now the $\mathcal{M}_r$ cylinders break “above” the $_{k-r}\mathcal{R}^1_{d+1,cpct}$ (equipped with $\check{}$, $\hat{}$, or S^1 decoration) discs instead of “below.” It is convenient in this model to reverse the labeling of the auxiliary points $p_1, \ldots, p_k$ so that now $0 < |p_k| \leq \cdots \leq |p_1| \leq \frac{1}{2}$ (so that their order is compatible with the ordering of auxiliary points on $\mathcal{M}_r$ moduli spaces).

Equipping these moduli spaces with perturbation data, counting solutions with sign twists as before, etc. defines the terms in the above pre-morphism exactly as in the previous subsections (with identical analysis to show that, for instance, the operation corresponding to ${}_k\mathcal{R}^{S^1}_{d,cpct}$ is the operation corresponding to ${}_{k-1}\hat{\mathcal{R}}^1_{d,cpct}$ composed with the Connes' B operator).

6. Calabi-Yau structures

6.1. The proper Calabi-Yau structure on the Fukaya category. We say an A_∞ category $\mathcal{A}$ is *proper* (sometimes called *compact*) if its cohomological morphism spaces $H^*(\hom_{\mathcal{A}}(X,Y))$ have total finite rank over $\mathbf{k}$, for each X, Y. Recall that for any object $X \in \mathcal{A}$, the inclusion of chain complexes $\hom(X,X) \to \mathrm{CH}_*(\mathcal{A})$ induces a chain map $[i] : H^*(\hom(X,X)) \to \mathrm{HH}_*(\mathcal{A})$.

DEFINITION 28. *Let $\mathcal{A}$ be a proper category. A chain map $tr : \mathrm{CH}_{*+n}(\mathcal{A}) \to \mathbf{k}$ is called a* weak proper Calabi-Yau structure*, or* non-degenerate trace *of dimension n if, for any two objects $X, Y \in \mathrm{ob}\,\mathcal{A}$, the composition*

$$H^*(\hom_{\mathcal{A}}(X,Y)) \otimes H^{n-*}(\hom_{\mathcal{A}}(Y,X)) \stackrel{[\mu^2_{\mathcal{A}}]}{\to} H^n(\hom_{\mathcal{A}}(Y,Y)) \stackrel{[i]}{\to} \mathrm{HH}_n(\mathcal{A}) \stackrel{[tr]}{\to} \mathbf{k} \tag{6.1}$$

is a perfect pairing.

REMARK 43. In the symplectic literature, weak proper Calabi-Yau structures of dimension n are sometimes defined as bimodule quasi-isomorphisms $\mathcal{A}_\Delta \stackrel{\sim}{\to} \mathcal{A}^\vee[n]$, where $\mathcal{A}_\Delta$ denotes the *diagonal bimodule* and $\mathcal{A}^\vee$ the *linear dual diagonal bimodule* — see [**S5**, (12j)] and §6.2 for brief conventions on A_∞ bimodules (see also [**T1**]). To explain the relationship between this definition and the one above, which has sometimes been called a *weakly cyclic structure* or ∞*-inner product* [**T1, S10**], note that for any compact A_∞ category $\mathcal{A}$, there are quasi-isomorphisms (with explicit chain-level models)

$$(\mathrm{CH}_*(\mathcal{A}))^\vee = \mathrm{CH}^*(\mathcal{A}, \mathcal{A}^\vee) \stackrel{\sim}{\leftarrow} \hom_{\mathcal{A}\text{–}\mathcal{A}}(\mathcal{A}_\Delta, \mathcal{A}^\vee) \tag{6.2}$$

where $\hom_{\mathcal{A}\text{–}\mathcal{A}}$ denotes morphisms in the category of A_∞ bimodules (see e.g., [**S3**] or [**G1**]). Under this correspondence, non-degenerate morphisms from $\mathrm{HH}_*(\mathcal{A}) \to \mathbf{k}$ as defined above correspond precisely (cohomologically) to weak Calabi-Yau structures, e.g., those bimodule morphisms from $\mathcal{A}_\Delta$ to $\mathcal{A}^\vee$ which are cohomology isomorphisms.

Remember that the Hochschild chain complex of an A_∞ category $\mathcal{A}$ comes equipped with a natural chain map to the (positive) cyclic homology chain complex

$$pr : \mathrm{CH}^{nu}_*(\mathcal{A}) \to \mathrm{CC}^+_*(\mathcal{A})$$

modeled on the chain level by the map that sends $\alpha \mapsto \alpha \cdot u^0$, for $\alpha \in \mathrm{CH}^{nu}(\mathcal{A})$.

DEFINITION 29 (c.f. Kontsevich-Soibelman [**KS2**]). *A* (strong) proper Calabi-Yau structure of degree *n is a chain map*

$$\tilde{tr} : \mathrm{CC}^+_*(\mathcal{A}) \to \mathbf{k}[-n] \tag{6.3}$$

from the (positive) cyclic homology chain complex of $\mathcal{A}$ to $\mathbf{k}$ of degree $-n$, such that the induced map $tr = \tilde{tr} \circ pr : \mathrm{CH}_*(\mathcal{A}) \to \mathbf{k}[-n]$ is a weak proper Calabi-Yau structure.

Via the model for cyclic chains given as $\mathrm{CC}_*(\mathcal{A}) := (\mathrm{CH}^{nu}_*(\mathcal{A})((u))/u\mathrm{CH}^{nu}_*(\mathcal{A})[[u]], b + uB^{nu})$, such an element $\tilde{tr}$ takes the form

$$\tilde{tr} := \sum_{i=0}^{\infty} tr^k u^k \tag{6.4}$$

where

$$tr^k := (\check{tr}^k \oplus \hat{tr}^k) : \mathrm{CH}^{nu}_*(\mathcal{A}) \to \mathbf{k}[-n-2k]. \tag{6.5}$$

We now complete the sketch of Theorem 2 described in the Introduction: first, define the putative proper Calabi-Yau structure as the composition:

$$\tilde{tr} : \mathrm{HC}^+_*(\mathcal{F}) \stackrel{\widetilde{\mathcal{OC}}}{\to} H^{*+n}(M, \partial M) \otimes_{\mathbf{k}} \mathbf{k}((u))/u\mathbf{k}[[u]] \to \mathbf{k} \tag{6.6}$$

where the last map sends $PD(pt) \cdot u^0 \in H^{2n}(M, \partial M)$ to 1, and other elements to zero. Instead of using a C^2 small Hamiltonian to define the Floer complex computing $H^{*+n}(M, \partial M)$ (which we only did for simultaneous compatibility with the symplectic cohomology case), we can pass to a geometric cycle model for the map $\widetilde{\mathcal{OC}}$ (and hence the map $\tilde{tr}$) described in §5.6.1. The map $\tilde{tr}$ in particular involves counts of the moduli spaces described there where the interior marked point $\bar{z}_{out}$ is *unconstrainted*, e.g., ${}_k\check{\mathcal{P}}^1_{d+1}([M]; \vec{x})$, ${}_k\hat{\mathcal{P}}^1_{d+1}([M]; \vec{x})$, and ${}_k\mathcal{P}^{S^1}_{d+1}([M]; \vec{x})$; see Figure 6.

The following well-known Lemma verifies the non-degeneracy property of the map $\tilde{tr}$:

LEMMA 20 (see e.g., [**S5**, (12j)], [**S10**, Lemma 2.4])). *The corresponding morphism* $[tr]$: $\mathrm{HH}_{*+n}(\mathcal{F}, \mathcal{F}) \to \mathbf{k}$ *is a non-degenerate trace (or weak proper Calabi-Yau structure).*

PROOF. This is an immediate consequence of Poincaré duality in Lagrangian Floer cohomology, see the references cited above. As a brief sketch, note that $tr \circ \mu^2 : \hom(X, Y) \otimes \hom(Y, X) \to \mathbf{k}$ is chain homotopic (and hence equal in cohomology) to a chain map which counts holomorphic discs with an interior marked point satisfying an empty constraint, and two (positive) boundary asymptotics on p, q, with corresponding Lagrangian boundary on x and y. Via a further homotopy of Floer data, one can arrange that the generators of $\hom(X, Y)$ and $\hom(Y, X)$ are in bijection (for instance if one is built out of time-1 flowlines of H and one out of time-1 flowlines of $-H$), and the only such rigid discs are constant discs between p and the corresponding $p^\vee$. $\square$

PROOF OF THEOREM 2. The above discussion constructs $\tilde{tr}$ and Lemma 20 verifies non-degeneracy. $\square$

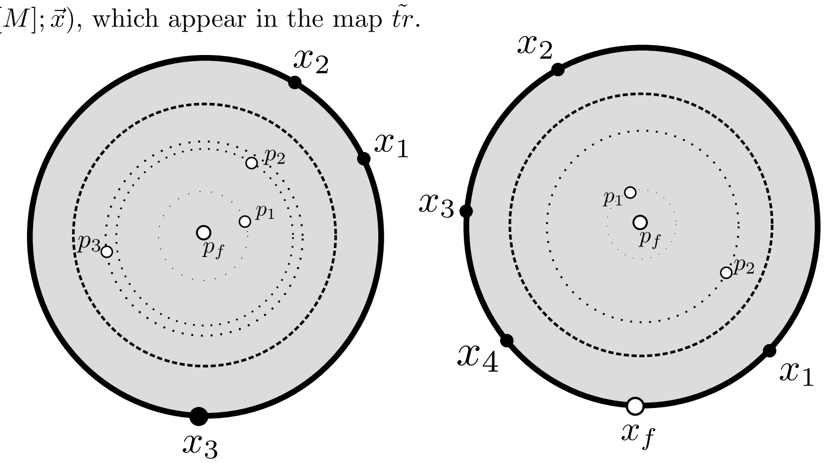


FIGURE 6. An image of representatives of moduli spaces ${}_3\check{\mathcal{P}}^1_3([M]; \vec{x})$ and ${}_2\hat{\mathcal{P}}^1_4([M]; \vec{x})$, which appear in the map $\tilde{tr}$.

6.2. The smooth Calabi-Yau structure on the Fukaya category. We give a brief overview of (a categorical version of) the notion of a *strong smooth Calabi-Yau structure*, following [**KV**]. Such structures are show in *loc. cit.* to induce chain-level topological field theory structures on the Hochschild chain complex of the given category, controlled by the open moduli space of curves with marked points and asymptotic markers. Crucially, not all operations are allowed; all moduli spaces appearing have at least *one output* (sometimes called a 'right positive field theory')

REMARK 44. Contrast this to the field theoretic operations associated to compact Calabi-Yau/cyclic A_∞ structures considered in [**C2, KS2**] where all operations must have at least *one input* (sometimes called a 'left positive' theory).

It will be necessary to make use of some of the theory of A_∞ *bimodules over a category* $\mathcal{C}$. We do so without much explanation, instead referring readers to existing references [**S3**, **T1**, **G1**]. Roughly, an A_∞ bimodule $\mathcal{P}$ is the data, of, for every pair of objects $\mathcal{C}$, a chain complex $(\mathcal{P}(X,Y), \mu^{0|1|0})$, along with 'higher multiplication maps' $\mu^{s|1|t} : \hom_{\mathcal{C}}(X_{s-1}, X_s) \otimes \cdots \otimes \hom_{\mathcal{C}}(X_0, X_1) \otimes \mathcal{P}(X_0, Y_t) \otimes \hom_{\mathcal{C}}(Y_{t-1}, Y_t) \otimes \cdots \otimes \hom_{\mathcal{C}}(Y_0, Y_1) \to \mathcal{P}(X_s, Y_0)$ satisfying a generalization of the A_∞ equations. A_∞ bimodules over $\mathcal{C}$ form a dg category $\mathcal{C}{-}\mathrm{mod}{-}\mathcal{C}$ with morphisms denoted $\hom^*_{\mathcal{C}\text{-}\mathcal{C}}(\mathcal{P}, \mathcal{Q})$ (for dg bimodules over a dg category, this chain complex corresponds to a particular chain model for the 'derived morphism space' using the bar resolution). The basic examples of bimodules we require are:

- the *diagonal bimodule* $\mathcal{C}_\Delta$ which associates to a pair of objects (K, L) the chain complex $\mathcal{C}_\Delta(K, L) := \hom_{\mathcal{C}}(L, K)$.
- for any pair of objects A, B, there is a *Yoneda bimodule* $\mathcal{Y}^l_A \otimes_{\mathbf{k}} \mathcal{Y}^r_B$ which associates to a pair of objects (K, L) the chain complex $\mathcal{Y}^l_A \otimes_{\mathbf{k}} \mathcal{Y}^r_B(K, L) := \hom_{\mathcal{C}}(A, K) \otimes \hom_{\mathcal{C}}(L, B)$.

Yoneda bimodules are the analogues of the free bimodule $A \otimes A^{op}$ in the category of bimodules over an associative algebra A (which are the same as $A \otimes A^{op}$ modules). Accordingly, we say a bimodule $\mathcal{P}$ *perfect* if, in the category $\mathcal{C}{-}\mathrm{mod}{-}\mathcal{C}$, it is *split-generated* by Yoneda bimodules. As mentioned in the introduction, we say that the category $\mathcal{C}$ is *(homologically) smooth* if $\mathcal{C}_\Delta$ is perfect. Recall for what follows that for any bimodule $\mathcal{P}$ there is a *cap product action*

$$\cap : \mathrm{HH}^*(\mathcal{C}, \mathcal{P}) \otimes \mathrm{HH}_*(\mathcal{C}, \mathcal{C}) \to \mathrm{HH}_*(\mathcal{C}, \mathcal{P}) \tag{6.7}$$

see e.g., [**G1**, §2.10] for chain level formulae; this is a slight variation on the formula presented there. (more generally, the cap products acts as $\mathrm{HH}^*(\mathcal{C}, \mathcal{P}) \otimes \mathrm{HH}_*(\mathcal{C}, \mathcal{Q}) \to \mathrm{HH}_*(\mathcal{C}, \mathcal{P} \otimes_{\mathcal{C}} \mathcal{Q})$).

Definition 30. *Let $\mathcal{C}$ be a homologically smooth A_∞ category. A cycle $\sigma \in \mathrm{CH}_{-n}(\mathcal{C}, \mathcal{C})$ is said to be a* non-degenerate co-trace*, or a* weak smooth Calabi-Yau structure*, if, for any objects K, L, the operation of capping with σ induces a (homological) isomorphism*

$$[\cap \sigma] : \mathrm{HH}^*(\mathcal{C}, \mathcal{Y}^l_K \otimes_{\mathbf{k}} \mathcal{Y}^r_L) \xrightarrow{\cong} \mathrm{HH}_{*-n}(\mathcal{C}, \mathcal{Y}^l_K \otimes_{\mathbf{k}} \mathcal{Y}^r_L) \simeq H^*(\hom_{\mathcal{C}}(K, L)). \tag{6.8}$$

Remark 45. The second isomorphism $\mathrm{HH}_{*-n}(\mathcal{C}, \mathcal{Y}^l_K \otimes_{\mathbf{k}} \mathcal{Y}^r_L) \simeq H^*(\hom_{\mathcal{C}}(K, L))$ always holds, at least for cohomologically unital categories, which we are always implicitly working with; the content is in the first.

Remark 46. Continuing Remark 43, there is an alternate perspective on Definition 30 using bimodules. Namely, for any bimodule $\mathcal{P}$, there is a naturally associated *bimodule dual* $\mathcal{P}^!$, defined for a pair of objects (K, L) as the chain complex $\mathcal{P}^!(K, L) := \hom^*_{\mathcal{C}\text{-}\mathcal{C}}(\mathcal{P}, \mathcal{Y}^l_K \otimes_{\mathbf{k}} \mathcal{Y}^r_L)$ (the higher bimodule structure is defined in [**G1**, Def. 2.40]; it is an A_∞ analogue of defining, for an A bimodule B, $B^! := \mathrm{RHom}_{A \otimes A^{op}}(B, A \otimes A^{op})$ where RHom is taken with respect to the outer bimodule structure on $A \otimes A^{op}$ and the bimodule structure on $B^!$ comes from the inner bimodule structure; see e.g., [**G5**, §20.5].

We abbreviate $\mathcal{C}^! := \mathcal{C}^!_\Delta$, and call $\mathcal{C}^!$ the *inverse dualizing bimodule*, following [**KS2**] (observe $\mathcal{C}^!(K, L) \simeq \mathrm{CH}^*(\mathcal{C}, \mathcal{Y}^l_K \otimes_{\mathbf{k}} \mathcal{Y}^r_L)$). For a homologically smooth category $\mathcal{C}$ one notes that there is a quasi-isomorphism $\mathrm{CH}_{-n}(\mathcal{C}) \simeq \hom^*_{\mathcal{C}\text{-}\mathcal{C}}(\mathcal{C}^!_\Delta, \mathcal{C}_\Delta)$ (see [**KS2**, Rmk. 8.2.4] for the case of A_∞ algebras), where the equivalence associates to any element, the bimodule morphism whose cohomology level map is the cap product operation (6.8). Non-degenerate cotraces correspond precisely then to bimodule quasi-isomorphisms $\mathcal{C}^! \xrightarrow{\sim} \mathcal{C}_\Delta$. Details of this will in the A_∞ categorical setting will be discussed more in [**CG**].

Let $\iota : \mathrm{CC}^-_*(\mathcal{C}) \to \mathrm{CH}_*(\mathcal{C})$ denote the chain-level model of the inclusion of homotopy fixed points mentioned in the introduction; concretely this is the chain map sending $\sum_{i=0}^\infty \alpha_i u^0 \mapsto \alpha_0$.

Definition 31. *Let $\mathcal{C}$ be a homologically smooth A_∞ category. A* (strong) smooth Calabi-Yau structure *is a cycle $\tilde{\sigma} \in \mathrm{CC}^-_{-n}(\mathcal{C})$ such that the corresponding element $\iota(\tilde{\sigma}) \in \mathrm{CH}_{-n}(\mathcal{C})$ is a weak smooth Calabi-Yau structure.*

Using these definitions and the cyclic open-closed map, we restate and prove Theorem 3:

THEOREM 6 (Theorem 3 above). *Suppose a Liouville manifold is* non-degenerate *in the sense of* [**G1**], *meaning that the map* $[\mathcal{OC}] : \mathrm{HH}_{*-n}(\mathcal{W}) \to SH^*(M)$ *hits 1. Then, the wrapped Fukaya category possesses a canonical* strong smooth Calabi-Yau structure*.*

PROOF. In [**G1**] it was proven that, assuming non-degeneracy of M, the map $[\mathcal{OC}] : \mathrm{HH}_{*-n}(\mathcal{W}) \to SH^*(M)$ is an isomorphism, $\mathcal{W}$ is homologically smooth, and moreover that the pre-image $[\sigma]$ of 1 gives a *weak smooth Calabi-Yau structure* in the sense described above. Corollary 1 implies that there is a commutative diagram of isomorphisms:

$$\begin{tikzcd} \mathrm{HC}^-_{*-n}(\mathcal{W}) \arrow[r, "\iota"'] \arrow[d, "{[\widetilde{\mathcal{OC}}^-]}"'] & \mathrm{HH}_{*-n}(\mathcal{W}) \arrow[d, "{[\mathcal{OC}]}"] \\ H^*(SC^*(M)^{hS^1}) \arrow[r, "\iota"'] & SH^*(M) \end{tikzcd} , \tag{6.9}$$

where the horizontal maps ι are the canonical maps $P^{hS^1} \to P$, $\sum_{i=0}^{\infty} \alpha_i u^i \mapsto \alpha_0$ modeling the *inclusion of homotopy fixed points.*

Recall that $1 \in SH^*(M)$ is defined to be the image of 1 under the natural map $H^*(M) \to SH^*(M)$. But this chain level map, which comes formally from the inclusion of constant loops in the free loop space, naturally extends to a morphism of A_∞ $C_{-*}(S^1)$ modules, using the trivial S^1 action on $C^*(M)$ (In particular, we can make choices so that $C^*(M)$ is a subcomplex of $SC^*(M)$ with trivial BV operators, see Lemma 11, c.f., [**ACF**, **Z**]). In particular, on the level of homotopy fixed points, there is a map

$$i : H^*(M)[u] \to H^*(SC^*(M)^{hS^1})$$

and hence there is a canonical lift $\tilde{1} = i(1 \cdot u^0) \in H^*(SC^*(M)^{hS^1})$ of the element $1 \in SH^*(M)$. Since $[\widetilde{\mathcal{OC}}^-]$ is an isomorphism, it follows that there is a unique element $\widetilde{\sigma} \in \mathrm{HC}^-_{*-n}(\mathcal{W})$ hitting $\tilde{1}$. By (6.9), $\iota(\widetilde{\sigma}) = \sigma$, providing the desired lift of the weak smooth Calabi-Yau structure. □

Appendix A. Moduli spaces and operations

A.1. A real blow-up of Deligne-Mumford space. We review, in a special case, the compactifications of moduli spaces of surfaces where some interior marked points are equipped with asymptotic markers, which are a real blow-up of Deligne-Mumford moduli space as constructed in [**KSV**]. In particular, we show how the abstract compactifications in the sense of [**KSV**] can be identified some of the specific models for the relevant strata we use in Section 5. (see also [**SS**] for an appearance in Floer cohomology)

To begin, let

$$\mathcal{M}_{2,0} \tag{A.1}$$

denote the space of spheres with 2 marked points z_1, z_2 removed and asymptotic markers τ_1, τ_2 around the z_1 and z_2, modulo automorphism. Fixing the position of z_1 and z_2 and one of τ_1 or τ_2 gives a diffeomorphism

$$\mathcal{M}_{2,0} \cong S^1.$$

On an arbitrary representative in $\mathcal{M}_{2,0}$, we can think of the map to S^1 as coming from the *difference in angles* between τ_1 and τ_2 (after, say, parallel transporting one tangent space the other along a geodesic path).

It is convenient to parametrize this difference by a point on the sphere itself, in the following manner (though this will break symmetry between z_1 and z_2). Let

$$\mathcal{M}_{2,1} \tag{A.2}$$

be the space of spheres with 2 marked points z_1, z_2 removed, an extra marked point y_1 and asymptotic markers τ_1, τ_2 around the z_1 and z_2, modulo automorphism such that, for any representative with position of z_1, z_2, and y fixed, τ_2 is pointing towards y. The remaining freedom in τ_1 once more gives a diffeomorphism $\mathcal{M}_{2,1} \cong S^1$.

We can take a different representative for elements of $\mathcal{M}_{2,1}$: up to biholomorphism any element of (A.2) is equal to a cylinder sending z_1 to $+\infty$, z_2 to $-\infty$) with fixed asymptotic direction around $+\infty$ and an extra marked point y at fixed height freely varying around S^1, such that the asymptotic marker at $-\infty$ coincides with the S^1 coordinate of y. Thus, we obtain an identification

$$\mathcal{M}_{2,1} \cong \mathcal{M}_1 \tag{A.3}$$

where $\mathcal{M}_1$ is the space in Definition 12.

Denote by

$$_s\mathcal{R}^1_k \tag{A.4}$$

the moduli space of discs $(S, z_1, \ldots, z_k, y, \tau_y, p_1, \ldots, p_s)$ with k boundary marked points $z_1, \ldots, z_k$ arranged in clockwise order, an interior marked point with asymptotic marker (y, τ_y), and interior marked points with no asymptotic markers $x_1, \ldots, x_s$, modulo autmorphism. Any element $\mathcal{R}^{k+1}_{1,s}$ admits a unique unit disc representative with z_k fixed at 1 and y at 0; call this the (z_k, y) *standard representative*. The positions of the asymptotic marker, remaining marked points, and interior marked points identify $\mathcal{R}^{k+1}$ with an open subset of $S^1 \times \mathbb{R}^{2s} \times \mathbb{R}^k$. The space (A.4) has another ordering-type constraint, on the positions of the interior marked points:

$$\text{Any element has standard representative satisfying } 0 < |p_1| < |p_2| < \cdots < |p_k|. \tag{A.5}$$

and a condition on the asymptotic marker:

$$\text{For the standard representative, } \tau_y \text{ points at } p_1. \tag{A.6}$$

The condition (A.5), which cuts out a manifold with corners of the space with p_i unconstrained, is technically convenient, as it reduces the types of bubbles that can occur with y). The compactification of interest, denoted

$$_s\overline{\mathcal{R}}^k_1 \tag{A.7}$$

differs from the Deligne-Mumford compactification in a couple respects: firstly, in the closures of the ordering conditions (A.5) away from zero, we allow points p_i and p_{i+1} to be coincident without bubbling off (alternatively, we can Deligne-Mumford compactify and collapse the relevant strata).

More interestingly, (A.7) is a real blow-up of the usual Deligne-Mumford compactification along any strata in which y and p_i points bubble off, in the following precise sense. Let $\Sigma = S_0 \cup_{y^+=y^-} S_1$ denote a nodal surface, where

- S_0 is a sphere containing interior marked points (y, τ_y), $p_1, \ldots, p_j$, and another marked point y^+, and
- S_1 is a disc with k boundary marked points $z_1, \ldots, z_k$, and interior marked points y^-, p_{j+1}, $\ldots$, p_s.

To occur as a possible degenerate limit of (A.4), the relevant points p_i on S_0 and S_1 must satisfy an ordering condition:

(A.8) For a representative S_0' of S_0 with y and y^+ at opposite poles, $0 < |p_1| < \cdots < |p_j| < |y^+|$, where $|p|$ denotes the geodesic distance from p to y for S_0'.

(A.9) For the (z_k, y^-) standard representative of S_1, $0 < |p_{j+1}| < \cdots < |p_s|$.

Also,

(A.10)

For the same representative S_0', the asymptotic marker τ_y should point (geodesically) towards p_1.

The relevant stratum of (A.7) consists of all such configurations $S_0 \cup_{y^+=y^-} S_1$ equipped with an extra piece of information: *a gluing angle*, which is a real positive line τ_{y^+,y^-} in $T_{y^+}S_0 \otimes T_{y^-}S_1$, or equivalently, a pair of asymptotic markers around y^+ and y^-, modulo the diagonal S^1 rotation action. Note that the set of gluing angles is S^1, giving such strata 1 higher codimension. We note that there is a natural choice of asymptotic marker on S_1, coming from the position of $|p_{j+1}|$ (or if $j = s$, the position of z_k) for the standard representative. Fixing such a choice identifies S_1 with an element of ${}_{s-j}\mathcal{R}_1^k$ and equips S_0 with a (freely varying) asymptotic marker τ_{y^+} at y^+. In a manner as in (A.3), we can identify S_0 up to biholomorphism with an element of $\mathcal{M}_j$. Hence, we've identified the relevant stratum with

$$\mathcal{M}_j \times {}_{s-j}\overline{\mathcal{R}}_1^k, \tag{A.11}$$

which will be useful in defining the relevant pseudoholomorphic curve counts.

A.2. Operations with a forgotten marked point. We introduce auxiliary degenerate operations that will arise as the codimension 1 boundary of the open-closed map and equivariant structure. This subsection is a very special case of the general discussion in [**G1**].

Let $d \geq 2$ and $i \in \{1, \ldots, d\}$. The *moduli space of discs with d marked points with ith boundary point forgotten*

$$\mathcal{R}^{d,f_i} \tag{A.12}$$

is exactly the moduli space of discs $\mathcal{R}^d$, with ith boundary marked point labeled as auxiliary.

The Deligne-Mumford compactification

$$\overline{\mathcal{R}}^{d,f_i} \tag{A.13}$$

is exactly the usual Deligne-Mumford compactification, along with the data of an *auxiliary label* at the relevant boundary marked point.

For $d > 2$, the *i-forgetful map*

$$\mathcal{F}_{d,i} : \mathcal{R}^{d,f_i} \to \mathcal{R}^{d-1}. \tag{A.14}$$

associates to a surface S the surface obtained by putting the ith point back in and forgetting it. This map admits an extension to the Deligne-Mumford compactification

$$\overline{\mathcal{F}}_{d,i} : \overline{\mathcal{R}}^{d,f_i} \to \overline{\mathcal{R}}^{d-1} \tag{A.15}$$

as follows: eliminate any non-main components with only one non-auxiliary marked point p, and label the positive marked point below this component by p. We say that any component not eliminated is *f-stable* and any component eliminated is *f-semistable*.

The above map is only well-defined for $d > 2$. In the semi-stable case $d = 2$, $\mathcal{R}^{2,f_i}$ is a point so one can define an ad hoc map

$$\mathcal{F}_i^{ss} : \mathcal{R}^{2,f_i} \to pt. \tag{A.16}$$

which associates to a surface S the (unstable) strip $\Sigma_1 = (-\infty, \infty) \times [0, 1]$ as follows: take the unique representative of S which, after its three marked points are removed, is biholomorphic to the strip Σ_1 with an additional puncture $(0, 0)$. Then, forget/put back in the point $(0, 0)$.

Definition 32. *A* forgotten Floer datum *for a stable disc with ith point auxiliary $S \in \overline{\mathcal{R}}^{d,f_i}$ consists of, for every component T of S,*

- *a Floer datum for T, if T does not contain the auxiliary point,*
- *a Floer datum for $\mathcal{F}_j(T)$, if T is f-stable and contains the auxiliary point as its jth input.*
- *A Floer datum on $\mathcal{F}_i^{ss}(T)$ which is* translation invariant, *if T is f-semistable.*

(by translation invariant, *we mean the following: note that Σ_1 has a canonical $\mathbb{R}$-action given by linear translation in the s coordinate. We require H, J, and the time-shifting map/weights to be invariant under this $\mathbb{R}$ action, and in particular should only depend on $t \in [0, 1]$ at most).*

In particular, this Floer datum should only depend on the point $\overline{\mathcal{F}}_{d,i}(S)$.

Proposition 19. *Let $i \in \{1, \ldots, d\}$ with $d > 1$. Then the operation associated to $\overline{\mathcal{R}}^{d,f_i}$ is zero if $d > 2$ and the identity operation $I(\cdot)$ (up to a sign) when $d = 2$.*

Sketch. Suppose first that $d > 2$, and let u be any solution to Floer's equation over the space $\mathcal{R}^{d,f_i}$ with domain S. Let Since the Floer data on S only depends on $\mathcal{F}_{d,i}(S)$, we see that maps from S' with $S' \in \mathcal{F}_{d,i}^{-1}(\mathcal{F}_{d,i}(S))$ also give solutions to Floer's equation with the same asymptotics. Moreover, the fibers of the map $\mathcal{F}_{d,i}$ are one-dimensional, implying that u cannot be rigid, and thus the associated operation is zero.

Now suppose that $d = 2$. Then the forgetful map associates to the single point $[S] \in \mathcal{R}^{2,f_i}$ the unstable strip with its translation invariant Floer datum. Since non-constant solutions can never be rigid (as, by translating, one can obtain other non-constant solutions), it follows that the only solutions are constant ones, and that the resulting operation is therefore the identity. □

References

[A] Mohammed Abouzaid, *A geometric criterion for generating the Fukaya category*, Publ. Math. Inst. Hautes Études Sci. **112** (2010), 191–240. MR2737980

[ACF] Peter Albers, Kai Cieliebak, and Urs Frauenfelder, *Symplectic tate homology* (201405), available at `1405.2303`.

[AFO+] M. Abouzaid, K. Fukaya, Y.-G. Oh, H. Ohta, and K. Ono, *Quantum cohomology and split generation in Lagrangian Floer theory.* in preparation.

[AS] Mohammed Abouzaid and Paul Seidel, *An open string analogue of Viterbo functoriality*, Geom. Topol. **14** (2010), no. 2, 627–718. MR2602848

[B] Dan Burghelea, *Cyclic homology and the algebraic K-theory of spaces. I*, Applications of algebraic K-theory to algebraic geometry and number theory, Part I, II (Boulder, Colo., 1983), 1986, pp. 89–115. MR862632 (88i:18009a)

[BEE] Frédéric Bourgeois, Tobias Ekholm, and Yakov Eliashberg, *Effect of Legendrian Surgery*, Geom. Topol. **16** (2012), no. 1, 301–389.

[BO] Frédéric Bourgeois and Alexandru Oancea, *S^1-equivariant symplectic homology and linearized contact homology*, Int. Math. Res. Not. (2016), doi:10.1093/imrn/rnw029, available at `http://arxiv.org/abs/1212.3731`.

[C1] Alain Connes, *Noncommutative differential geometry*, Inst. Hautes Études Sci. Publ. Math. **62** (1985), 257–360. MR823176 (87i:58162)

[C2] Kevin Costello, *Topological conformal field theories and Calabi-Yau categories*, Adv. Math. **210** (2007), no. 1, 165–214. MR2298823 (2008f:14071)

[C3] Kevin J. Costello, *The Gromov-Witten potential associated to a TCFT* (2005), available at `arXiv:math/0509264`.

[CFH] K. Cieliebak, A. Floer, and H. Hofer, *Symplectic homology. II. A general construction*, Math. Z. **218** (1995), no. 1, 103–122. MR1312580 (95m:58055)

[CFHW] K. Cieliebak, A. Floer, H. Hofer, and K. Wysocki, *Applications of symplectic homology. II. Stability of the action spectrum*, Math. Z. **223** (1996), no. 1, 27–45. MR1408861 (97j:58045)

[CG] Ralph Cohen and Sheel Ganatra, *Calabi-Yau categories, the Floer theory of the cotangent bundle, and the string topology of the base*, 2015. In preparation.

[CL] Cheol-Hyun Cho and Sangwook Lee, *Potentials of homotopy cyclic A_∞-algebras*, Homology Homotopy Appl. **14** (2012), no. 1, 203–220. MR2954673

[F1] Andreas Floer, *Witten's complex and infinite-dimensional Morse theory*, J. Differential Geom. **30** (1989), no. 1, 207–221. MR1001276 (90d:58029)

[F2] Kenji Fukaya, *Cyclic symmetry and adic convergence in Lagrangian Floer theory*, Kyoto J. Math. **50** (2010), no. 3, 521–590. MR2723862 (2011m:53169)

[F3] ———, *Counting pseudo-holomorphic discs in Calabi-Yau 3-folds*, Tohoku Math. J. (2) **63** (2011), no. 4, 697–727. MR2872962

[FH] A. Floer and H. Hofer, *Symplectic homology. I. Open sets in $\mathbf{C}^n$*, Math. Z. **215** (1994), no. 1, 37–88. MR1254813 (95b:58059)

[FOOO] Kenji Fukaya, Yong-Geun Oh, Hiroshi Ohta, and Kaoru Ono, *Lagrangian intersection Floer theory: anomaly and obstruction. Part II*, AMS/IP Studies in Advanced Mathematics, vol. 46, American Mathematical Society, Providence, RI, 2009. MR2548482

[FSS] K. Fukaya, P. Seidel, and I. Smith, *The symplectic geometry of cotangent bundles from a categorical viewpoint*, Homological mirror symmetry, 2009, pp. 1–26. MR2596633 (2011c:53213)

[G1] Sheel Ganatra, *Symplectic cohomology and duality for the wrapped Fukaya category*, Ph.D. Thesis, 2012.

[G2] ———, *Symplectic cohomology and duality for the wrapped Fukaya category*, 2012. Ph.D. Thesis, MIT. Available at `http://arXiv.org/abs/1304.7312`.

[G3] Sheel Ganatra, *Symplectic integral transforms from open-closed string maps*, 2015. Available at `http://www-bcf.usc.edu/~sheelgan/materials/wrapcy1.pdf`.

[G4] ———, *Automatically generating Fukaya categories and computing quantum cohomology*, 2016. available at http://arXiv.org/abs/1605.07702.

[G5] Victor Ginzburg, *Lectures on Noncommutative Geometry* (2005), available at arXiv:math/0506603.

[GPS1] Sheel Ganatra, Tim Perutz, and Nick Sheridan, *The cyclic open-closed map and noncommutative Hodge structures*. In preparation.

[GPS2] ———, *Mirror symmetry: from categories to curve counts*, 2015. available at arXiv.org/abs/1510.03839.

[K1] Christian Kassel, *Cyclic homology, comodules, and mixed complexes*, J. Algebra **107** (1987), no. 1, 195–216. MR883882 (88k:18019)

[K2] Bernhard Keller, *On the cyclic homology of exact categories*, J. Pure Appl. Algebra **136** (1999), no. 1, 1–56. MR1667558 (99m:18012)

[K3] ———, *A-infinity algebras, modules and functor categories*, Trends in representation theory of algebras and related topics, 2006, pp. 67–93. MR2258042

[K4] Maxim Kontsevich, *XI Solomon Lefschetz Memorial Lecture series: Hodge structures in non-commutative geometry*, Non-commutative geometry in mathematics and physics, 2008, pp. 1–21. Notes by Ernesto Lupercio. MR2444365 (2009m:53236)

[KS1] M. Kontsevich and Y. Soibelman, *Notes on A_∞-algebras, A_∞-categories and non-commutative geometry*, Homological mirror symmetry, 2009, pp. 153–219. MR2596638

[KS2] Maxim Kontsevich and Yan Soibelman, *Notes on A-infinity algebras, A-infinity categories and non-commutative geometry. I*, 2006. Available at arXiv.org/abs/math/0606241.

[KSV] Takashi Kimura, Jim Stasheff, and Alexander A. Voronov, *On operad structures of moduli spaces and string theory*, Comm. Math. Phys. **171** (1995), no. 1, 1–25. MR1341693 (96k:14019)

[KV] Maxim Kontsevich and Yannis Vlassopoulos. in preparation.

[L1] K. Lefevre, *Sur les a_∞-catégories*, Ph.D. Thesis, 2002.

[L2] Jean-Louis Loday, *Cyclic homology*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 301, Springer-Verlag, Berlin, 1992. Appendix E by María O. Ronco. MR1217970 (94a:19004)

[LQ] Jean-Louis Loday and Daniel Quillen, *Cyclic homology and the Lie algebra homology of matrices*, Comment. Math. Helv. **59** (1984), no. 4, 569–591. MR780077 (86i:17003)

[M1] Randy McCarthy, *The cyclic homology of an exact category*, J. Pure Appl. Algebra **93** (1994), no. 3, 251–296. MR1275967 (95b:19002)

[M2] John McCleary, *A user's guide to spectral sequences*, Second, Cambridge Studies in Advanced Mathematics, vol. 58, Cambridge University Press, Cambridge, 2001. MR1793722 (2002c:55027)

[M3] Luc Menichi, *String topology for spheres*, Comment. Math. Helv. **84** (2009), no. 1, 135–157. With an appendix by Gerald Gaudens and Menichi. MR2466078 (2009k:55017)

[PSS] S. Piunikhin, D. Salamon, and M. Schwarz, *Symplectic Floer-Donaldson theory and quantum cohomology*, Contact and symplectic geometry (Cambridge, 1994), 1996, pp. 171–200. MR1432464 (97m:57053)

[R] Alexander F. Ritter, *Topological quantum field theory structure on symplectic cohomology* (2010), available at arXiv:1003.1781.

[S1] Paul Seidel, *Graded Lagrangian submanifolds*, Bull. Soc. Math. France **128** (2000), no. 1, 103–149. MR1765826 (2001c:53114)

[S2] ———, *Fukaya categories and deformations*, Proceedings of the International Congress of Mathematicians, Vol. II (Beijing, 2002), 2002, pp. 351–360. MR1957046 (2004a:53110)

[S3] ———, *A_∞-subalgebras and natural transformations*, Homology, Homotopy Appl. **10** (2008), no. 2, 83–114. MR2426130 (2010k:53154)

[S4] ———, *A biased view of symplectic cohomology*, Current developments in mathematics, 2006, 2008, pp. 211–253. MR2459307 (2010k:53153)

[S5] ———, *Fukaya categories and Picard-Lefschetz theory*, Zurich Lectures in Advanced Mathematics, European Mathematical Society (EMS), Zürich, 2008. MR2441780 (2009f:53143)

[S6] ———, *Lectures on Categorical Dynamics and Symplectic Topology*, 2013. Available at http://math.mit.edu/~seidel/937/lecture-notes.pdf.

[S7] ———, *Disjoinable Lagrangian spheres and dilations*, Invent. Math. **197** (2014), no. 2, 299–359. MR3232008

[S8] ———, *Connections on equivariant hamiltonian floer cohomology*, 2016. Available at https://arxiv.org/abs/1612.07460.

[S9] Nick Sheridan, *Formulae in noncommutative Hodge theory*, 2015. available at http://arxiv.org/abs/1510.03795.

[S10] ———, *On the Fukaya category of a Fano hypersurface in projective space*, Publications mathématiques de l'IHÉS (2016), 1–153.

[SS] Paul Seidel and Jake P. Solomon, *Symplectic cohomology and q-intersection numbers*, Geom. Funct. Anal. **22** (2012), no. 2, 443–477. MR2929070

[T1] Thomas Tradler, *Infinity-inner-products on A-infinity-algebras*, J. Homotopy Relat. Struct. **3** (2008), no. 1, 245–271. MR2426181 (2010g:16016)

[T2] B. L. Tsygan, *Homology of matrix Lie algebras over rings and the Hochschild homology*, Uspekhi Mat. Nauk **38** (1983), no. 2(230), 217–218. MR695483 (85i:17014)

[V] C. Viterbo, *Functors and computations in Floer homology with applications. I*, Geom. Funct. Anal. **9** (1999), no. 5, 985–1033. MR1726235 (2000j:53115)

[Z] Jingyu Zhao, *Periodic symplectic cohomologies*, 2014. Available at `http://arxiv.org/abs/1405.2084`.