GENERATING FUKAYA CATEGORIES OF LG MODELS
DRAFT

MOHAMMED ABOUZAID AND SHEEL GANATRA

Abstract. The Fukaya category of a Landau-Ginzburg model (E, W) enlarges the Fukaya category of E by including certain non-compact Lagrangians with asymmetric perturbations at infinity involving W. We give a criterion for when a finite collection of Lagrangians split-generates this Fukaya category, in the spirit of [A, AFO$^+$]. This criterion differs even formally from previous criteria, thanks to new structures that have no analogue in previous settings, most notably the existence of a (geometric) Serre functor. Other notable ingredients are a new closed string Floer homology ring $HF^*(E, W)$ associated to the pair (E, W), and operations between this ring and the Fukaya category of (E, W). As a first application we show (under suitable hypotheses) that the the derived Fukaya category is preserved by quadratic stabilization, verifying a proposal of Kontsevich.

1. Introduction

Mirror symmetry is well known to extend as a phenomenon beyond the Calabi-Yau setting @@CITE, although the formulation then becomes asymmetric: for instance, the mirror to a compact *Fano* (or general type) Kähler manifold X is expected to be a *Landau-Ginzburg (LG) model*: a pair $(\check{X}, W)$ of a non-compact Calabi-Yau equipped with a holomorphic function $W : \check{X} \to \mathbb{C}$ called a *superpotential.* Putting aside the interesting questions of how to construct such a mirror $(\check{X}, W)$ given X (see e.g., [HV, A, AAK]), the mirror symmetry prediction, in its various forms, then says that the complex or symplectic geometry of X is reflected in the *symplectic* or *complex* geometry of the *singularities of* W *on* $\check{X}$. In particular, *Homological Mirror Symmetry* in the sense of Kontsevich [K1] predicts equivalences of categories:

$$D^\pi \mathcal{F}(X) \cong D^b \mathrm{Sing}(\check{X}, W) \tag{1.1}$$

$$D^b \mathrm{Coh}(X) \cong D^\pi \mathcal{F}(\check{X}, W). \tag{1.2}$$

where $D^\pi \mathcal{F}(X)$ denotes the *split-closed derived Fukaya category*, and $D^b\mathrm{Coh}(X)$ and $D^b\mathrm{Sing}(\check{X}, W)$ denote the triangulated categories of coherent sheaves and *singularities* respectively (this latter category has been defined by Orlov [**?Orlov:ys**], and has an equivalent model in some cases known called the category of *matrix factorizations*).

This (entirely symplectic) paper is concerned with the remaining category associated to the second equivalence (1.2), which is called the *Fukaya category of the Landau-Ginzburg model* $\mathcal{F}(\check{X}, W)$. The idea that there should be such a Fukaya category is due to Kontsevich [K2]. Though this category has been studied quite extensively by Seidel when W is a *symplectic Lefschetz fibration* [S1, S2, S1, S5, S6] (in which case the category is often called the *Fukaya-Seidel category*), this category has been only briefly developed for general W, aside from a first general construction that has been given by Abouzaid-Seidel [AS], and other work in progress [AA]. There is good reason to do so: from a mirror symmetry perspective, more general pairs $(\check{X}, W)$ are already known to arise in the geometric construction of mirrors to hypersurfaces in toric varieties [AAK]. From a symplectic standpoint, the Fukaya categories of pairs $(\check{X}, W)$ (partly conjecturally) act as an intermediate object which helps to identify the effect of the monodromy symplectomorphisms μ on the Fukaya category of the fibre, as well as the fixed point Floer cohomology of μ, see e.g., [?Seidel:2001fk] and recent work of Abouzaid-Smith [?Abouzaid:2015ac] (which in turn makes use of a special case of the work developed in the sequel to our article).

Our main result is a *generation criterion* for computing the Fukaya category of a Landau-Ginzburg model up to Morita equivalence, in the sense of generation criteria which have appeared in other Fukaya categorical contexts [A, ?AF000], see Theorem 14 below. This criterion differs even formally in several ways from the statements that appear in [A, ?AF000], thanks to additional input from asymmetric geometric structures that have no analogue in those settings, most notably the existence of a non-trivial geometric *Serre functor* [?Seidel:2009dq, S4].

We apply our result to give a first new computation, under suitable hypotheses, of the (derived) Fukaya categories of LG models with a single Morse-Bott critical component. In particular, under these hypotheses we verify a proposal of Kontsevich [K3] (also described in detail in [AAK]) that the Fukaya category behaves in an expected manner after (dimensional) 'stabilization', a process which passes through Landau-Ginzubrg models.

A detailed discussion follows.

1.1. **Fukaya categories of Landau-Ginzburg (LG) Models.**

Impressionistically, a *symplectic Landau-Ginzburg (LG) model* is a pair

$$(E^{2n}, W)$$

consisting of a non-compact symplectic manifold E equipped with a map

$$W : E \to \mathbb{C}$$

which is a genuine symplectic fibration away from a compact region $K \subset \mathbb{C}$, see §@@REF for our precise setup. As in [S2, §15], the particular hypotheses imposed guarantee that symplectic parallel transport maps are well-defined, and in particular symplectically identify the general fibers of W; which we unambiguously refer to as M. Symplectic parallel transport around a big (positively oriented) loop in the base containing $\mathbb{C}$ induces a *monodromy symplectomorphism*

$$\mu : M \to M, \tag{1.3}$$

unique up to Hamiltonian isotopy.

Remark 1. From the point of view of having simple pseudoholomorphic curve theory, we impose standard technical hypotheses (such as monotonicity or exactness) on E.

Example 2 (Lefschetz-(Bott) fibrations). The simplest example of such a pair (E, W) is the case of a *symplectic Lefschetz-Bott fibration*; this is the case that the compact set K above is a union of points $\{p_i\}$, and the critical locus of W is a union of disjoint symplectic submanifolds $\cup\{M_i^{crit}\}$ (without loss of generality projecting to distinct critical values $p_i = W(M_i^{crit})$), with W satisfying a Morse-like non-degeneracy condition in directions normal to Z_i, e.g., the holomorphic local model at a point in Z_i is $(z_1, \ldots, z_n) \mapsto \sum_{i=1}^k z_i^2$ (and additional mostly technical hypotheses— see @@REF). A *symplectic Lefschetz fibration* arises as the special case that each critical submanifold M_i^{crit} is a point, just denoted m_i.

Example 3 (More degenerate singularities). The following example arises in mirror symmetry as a mirror candidate for the pair of pants $\mathbb{P}^1\backslash\{0,1,\infty\}$: $(E,W) := (\mathbb{C}^3, -xyz)$ [S3, AAK]. In this (and many other examples described in [AAK]), the singular locus is no longer a smooth manifold but rather has *stratified symplectic structure.*

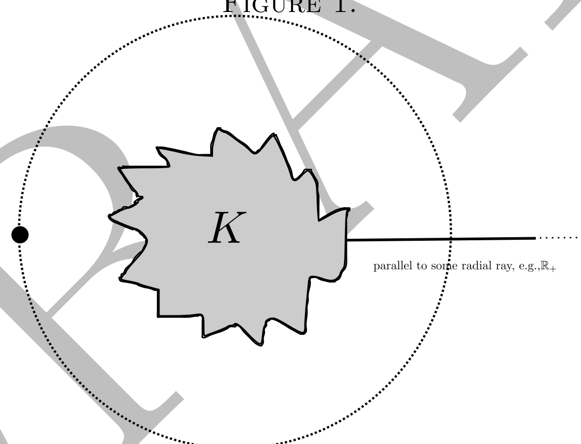
The *Fukaya category of an LG model* (E,W)

$$\mathcal{F}(E,W)$$

enlarges the Fukaya category of compact Lagrangians by allowing certain non-compact Lagrangians which have a specified behavior with respect to the map W (again, we will impose technical constraints such as exactness, monotonicity, etc., in order to simplify the holomorphic curve theory). Specifically, if L is non-compact, it should be properly embedded, fibrewise compact for W, and outside a compact set, project under W to a radial ray (or collection of radial rays) *disjoint from the negative real axis (or angle $-\pi$).*

See Figure 1 for a picture of the allowed image of such a Lagrangian. Crucially for later applications, our allowed Lagrangians can have multiple ends (this is the main fashion in which we generalize [AS]). We denote by A_L... this defines a partial ordering on Lagrangians via the rule:

FIGURE 1.

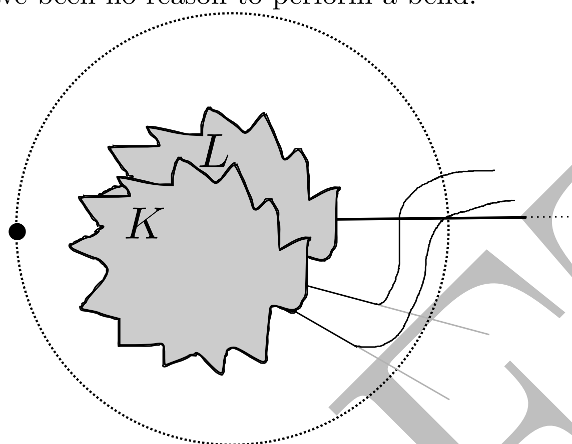


In order to obtain well-defined groups, the cohomology level morphism spaces are computed via the following ('asymmetric near ∞') perturbation, a suggestion of which goes back to ideas of Konstevich [K2]. One defines

$$H^*(\hom_{\mathcal{F}(E,W)}(K,L)) := HF^*(\phi_\epsilon(K), L), \tag{1.4}$$

where ϕ_ϵ denotes the lift to E of a Hamiltonian flow on $\mathbb{C}$ which agrees outside a compact set with a 'time ϵ positive bend'; here ϵ should be sufficiently large so that all the ends of K are pushed "above" L (but not so far that they wrap past the forbidden angle); see Figure 2. Also, ϕ_ϵ may further involve some compactly supported Hamiltonian perturbations in E; we ignore these here as they are unproblematic. Note that the resulting Floer cohomology is independent of such ϵ.

FIGURE 2. Bending the ends of (the image of) K past L to compute $H^*(\hom_{\mathcal{F}}(K, L))$. Note that if we had been computing $H^*(\hom_{\mathcal{F}}(L, K))$ instead, the ends of L would have already been above the ends of K, hence there would have been no reason to perform a bend.



Remark 4. Strictly speaking, if a Lagrangian has multiple ends, there may not exist a 'bend' large enough to push all of the ends of K past L. One could instead make use of the flow of a Hamiltonian that is *cut off* to 0 near angle -2π. This has the effect of simultaneously pushing the ends together and bending then towards angle $-\pi$. In order to set up higher compositions in the Fukaya category, we use an alternate (but presumably equivalent) method. We omit these issues from the introduction.

Remark 5. We expect a relationship of our methods to the the work of Biran-Cornea on *Lagrangian cobordism* [BC1, BC2, BC3], which also involves Fukaya categories of non-compact Lagrangians with multiple ends.

Remark 6. @@Insert citations to Kontsevich and Seidel for alternate models. Make a remark that some models of this category have all Lagrangians asymptotic to the same single direction near infinity, and then morphisms involve only very small perturbations. This is an equivalent category.

In order to define the cohomology composition $H^*(\hom_{\mathcal{F}}(L_1, L_2)) \otimes H^*(\hom_{\mathcal{F}}(L_0, L_1)) \to H^*(\hom_{\mathcal{F}}(L_0, L_2))$, we choose a pair of numbers ϵ_0, ϵ_1 so that $\phi_{\epsilon_0} L_0 > \phi_{\epsilon_1} L_1 > L_2$, and then count usual holomorphic triangles

$$HF^*(\phi_{\epsilon_1} L_1, L_2) \otimes HF^*(\phi_{\epsilon_0} L_0, \phi_{\epsilon_1} L_1) \to HF^*(\phi_{\epsilon_0} L_0, L_2). \tag{1.5}$$

the second group can be also identified with $H^*(\hom(L_0, L_1))$, thus giving the desired composition.

The basic challenge in defining chain level structures is to find a method of identifying the chain complexes appearing in such perturbations near ∞, e.g., morphisms may have been defined between L_1 and L_2 using a bend by ϵ', but (1.5) may require using a smaller ϵ_1. The issue is particularly delicate because of compactness properties of moduli spaces with non-compact perturbation terms in such identifications.

We make use of the solution to these issues developed in [AS]. In this procedure, an auxiliary 'directed category' $\mathcal{O}$ is defined, which does not involve any perturbations at ∞ (but mandates that the morphisms from L_1 and L_2 are zero unless one already knows

$L_1 > L_2$). The desired category $\mathcal{F}$ is then constructed via an algebraic 'localization' process, in which L_1 is essentially forced to become isomorphic to $\phi_\epsilon L_1$ (or more specifically, certain distinguished morphisms between $\phi_\epsilon L_1$ and L_1, called *quasi-units*, are forced to become isomorphisms). In this category, morphisms spaces a priori look rather different than (1.4), and the fact that they agree cohomologically (and with (1.5)) is then a result (which we call the 'correct position Lemma', see e.g., Lemma @@REF). We review this construction along with an extension to Lagrangians with multiple ends in @@REF below.

1.2. **Geometric structures.** Our first construction associates to the pair (E, W) a (closed string, or Hamiltonian) Floer cohomology group which is an invariant of the pair (E, W), and which we expect to correspond to the Hochschild cohomology of the Fukaya category of the LG model, by a conjecture of Seidel's:

Proposition 7 (compare [?Seidel:2001fk], Conjecture 6.1)**.** *There exists a Floer cohomology group $HF^*(E, W)$, fitting into an exact triangle*

$$\begin{tikzcd} H^*(E) \arrow[rr] & & HF^*(E,W) \arrow[dl] \\ & HF^*(\mu) \arrow[ul, "{[1]}"] & \end{tikzcd} \tag{1.6}$$

where $HF^*(\mu)$ denotes the fixed point Floer cohomology of μ (1.3) *In particular, there is a distinguished element* 1 *for $HF^*(E, W)$. Moreover, there is an algebra structure on $HF^*(E, W)$ for which the element* 1 *is a unit.*

Remark 8. An alternate but isomorphic model for $HF^*(E, W)$ as a group has been recently proposed by Seidel [S5].

Remark 9. @@ say something about the rather difference between this group and the Floer homology of a 'once-wrapped radial Hamiltonian.' (as a chain complex, instead of an S^1 worth of orbits corresponding to a loop around the base, this has only half of the relevant orbits. In particular, this group cannot have a BV operator, which aligns with what we understand from mirror symmetry: it is mirror to a *non-Calabi-Yau variety*, and polyvectorfields on non-Calabi-yau varieties do not have BV operators.

We expect that $HF^*(E, W)$ should be an interesting symplectic invariant in its own right of a symplectic LG model. As further evidence that it is the 'right' group, we show it has some of the expected compatibilities with $\mathcal{F}(E, W)$; for instance:

Lemma 10. *For any object K, there exists a geometric map*

$$[\mathcal{CO}] : HF^*(E, W) \to \mathrm{H}^*(\hom_{\mathcal{F}}(K, K)), \tag{1.7}$$

sending 1 to the identity $[id_K]$.

Indeed, our methods should give a map from $HF^*(E, W)$ to the Hochschild cohomology $\mathrm{HH}^*(\mathcal{F}(E, W), \mathcal{F}(E, W))$.

There is also a map from Lagrangian Floer cohomology (and a version of Hochschild *homology*) to $HF^*(E, W)$ as well, but to introduce it we will need to describe an extra operation: Let $\phi_{2\pi}$ denote the (lift to E of a) Hamiltonian flow on $\mathbb{C}$ that is the identity in a compact set and agrees with a '2π bend' near ∞: see Figure @@REF. This operation sends admissible Lagrangians to admissible Lagrangians, and in particular should induce an automorphism

$$\phi_{2\pi} : \mathcal{F}(E, W) \to \mathcal{F}(E, W). \tag{1.8}$$

Conjecture 11 (Kontsevich, Seidel @@INSERT-CITATION)**.** *$\phi_{2\pi}$ should agree with the* inverse to the Serre functor *for the category $\mathcal{F}(E, W)$.*

Recall that a bimodule $\mathcal{B}$ over an A_∞ category $\mathcal{C}$, is, to first order, an assignment of a chain complex $\mathcal{B}(X, Y)$ for every pair of objects X, Y in $\mathcal{C}$, together with 'left and right multiplications' $\hom_{\mathcal{C}}(X_0, X_1) \otimes \mathcal{B}(X_0, Y) \to \mathcal{B}(X_1, Y)$, $\mathcal{B}(X, Y_0) \otimes \hom_{\mathcal{C}}(Y_1, Y_0) \to \mathcal{B}(X, Y_1)$ and higher multiplications $\mu^{k|1|l}$ described below in @@REF.

Any automorphism of a category induces a bimodule, called its *graph*; in particular from (1.8) we have:

Proposition 12. *There exists a $\mathcal{F}$-bimodule, denoted $\mathcal{B}_{\phi_{2\pi}}$, satisfying $\mathcal{B}_{\phi_{2\pi}}(A, B) = \hom_{\mathcal{F}}(\phi_{2\pi} A, B)$, for $A, B \in \operatorname{ob} \mathcal{F}$.*

Recall that the *Hochschild homology* $\mathrm{HH}_*(\mathcal{C}, \mathcal{B})$ of a category $\mathcal{C}$ with coefficients in a bimodule $\mathcal{B}$ is the homology of a chain complex, which, explicitly is the direct sum of for any k and any $k+1$-tuple of objects $X_0, \ldots, X_k \in \operatorname{ob} \mathcal{A}$, the vector spaces

$$\mathcal{B}(X_k, X_0) \otimes \hom_{\mathcal{C}}(X_{k-1}, X_k) \otimes \cdots \otimes \hom_{\mathcal{C}}(X_0, X_1); \tag{1.9}$$

we refer to the underlying chain complex as $\mathrm{CC}_*(\mathcal{C}, \mathcal{B})$ (see @@REF for the differential, which involves all of the A_∞ and bimodule structure maps, grading etc.)

Using the bimodule $\mathcal{B}_{\phi_{2\pi}}$, we construct an map to the Floer cohomology $HF^*(E, W)$.

Proposition 13. *There is a geometric map $[\mathcal{OC}] : \mathrm{HH}_*(\mathcal{F}, \mathcal{B}_{\phi_{2\pi}}) \to HF^*(E, W)$.*

To lowest order, for any Lagrangian K, this is a map

$$HF^*(\phi_{2\pi+\epsilon} K, K) \to HF^*(E, W), \tag{1.10}$$

which is rather interestingly asymmetric to (1.7) (unlike such structures that have appeared for the Fukaya category of compact Lagrangians and the wrapped Fukaya categories).

1.3. Main Result.

With most of the structures established, we can state our main Theorem:

Theorem 14. *Let $\mathcal{A} \subset \mathcal{F}$ be a full subcategory of $\mathcal{F}(E, W)$. If the restriction of the open-closed map*

$$[\mathcal{OC}]|_{\mathcal{A}} : \mathrm{HH}_*(\mathcal{A}, \mathcal{B}_{\phi_{2\pi}}|_{\mathcal{A}}) \to HF^*(E, W) \tag{1.11}$$

has $1 \in HF^*(E, W)$ in its image, then the objects of $\mathcal{A}$ split-generate $\mathcal{F}$.

Remark 15. @@Insert more detailed remark about contrast to existing generation criteria.

The proof of Theorem 14, as in [A], involves applying version of the *Cardy condition* in topological field theory, see Proposition 16 below. There is one additional new input, which takes the form of a geometric 'coproduct map,' which to first order is a map

$$H^* \hom_{\mathcal{F}}(\phi_{2\pi+\epsilon} L, L) \to H^* \hom_{\mathcal{F}}(K, L) \otimes H^* \hom_{\mathcal{F}}(L, K)$$

for any K, and more generally gives a morphism of bimodules

$$\mathcal{B}_{\phi_{2\pi}}|_{\mathcal{A}} \to \mathcal{Y}^l_K \otimes_{\mathbb{K}} \mathcal{Y}^r_K.$$

where $\mathcal{Y}^l_K \otimes_{\mathbb{K}} \mathcal{Y}^r_K$ is a *Yoneda bimodule*, defined on a pair of objects (A, B) as the chain complex $\mathcal{Y}^l_K \otimes_{\mathbb{K}} \mathcal{Y}^r_K(A, B) := \hom_{\mathcal{F}}(K, A) \otimes \hom_{\mathcal{F}}(B, K)$.

Proposition 16 (Cardy condition)**.** *For any subcategory $\mathcal{A} \subset \mathcal{F}(E, W)$, and any object $K \in \mathrm{ob}\, \mathcal{F}(E, W)$ there is a morphism of bimodules $\Delta_{\mathcal{F}} : \mathcal{B}_{\phi_{2\pi}}|_{\mathcal{A}} \to \mathcal{Y}^l_K \otimes_{\mathbb{K}} \mathcal{Y}^r_K$ fitting into a commutative diagram*

$$\begin{tikzcd} \mathrm{HH}_*(\mathcal{A}, \mathcal{B}_{\phi_{2\pi}}|_{\mathcal{A}}) \arrow[r, "(\Delta_{\mathcal{F}})_*"] \arrow[d, "{[\mathcal{OC}_{\mathcal{F}}]}"] & H^*(\mathcal{Y}^r_K \otimes_{\mathcal{A}} \mathcal{Y}^l_K) \arrow[d, "\mu_{\mathcal{F}}"] \\ HF^*(E, W) \arrow[r, "{[\mathcal{CO}_{\mathcal{F}}]}"] & H^*(\hom_{\mathcal{F}}(K, K)) \end{tikzcd} , \tag{1.12}$$

Proof of Theorem 14. By all of the results above, if $[\mathcal{OC}]|_{\mathcal{A}} : \mathrm{HH}_*(\mathcal{A}, \mathcal{B}_{\phi_{2\pi}}|_{\mathcal{A}}) \to HF^*(E, W)$ hits 1, since the map $\mathcal{CO}_{\mathcal{F}}$ sends 1 to 1 for any K, it follows that for any $K \in \mathcal{F}$, the map $[\mu_{\mathcal{F}}] : H^*(\mathcal{Y}^r_K \otimes_{\mathcal{A}} \mathcal{Y}^l_K) \to H^*(\hom_{\mathcal{F}}(K, K))$ hits 1. But this is known to hold by @@CITE-DRINFELD-ABOUZAID if and only if $\mathcal{A}$ split-generates the object K. $\square$

@@explain algebraic criterion for split-generation more.

1.4. First Applications.

1.4.1. *Morse-Bott fibrations with one critical component.* We restrict to the special case where (E, W) is a Lefschetz-Bott fibration in the sense of Example 2, and assume further that (E, W) has a single critical component M^{crit}, which is a symplectic submanifold of E and without loss of generality lives above $W = 0$. Let $\gamma : [0, 1) \to \mathbb{C}$ be a path from 0 to ∞ which is a radial ray near ∞ and let $L \subset M^{crit}$ be a Lagrangian. For any such pair, we obtain a *generalized thimble*

$$\Delta^{\gamma}_L \in \mathcal{F}(E, W)$$

defined to be the set of points in E that parallel transport along γ to L; this is an $\mathbb{R}^k$ bundle over L (here, k is the complex codimension of M^{crit}.)

If L_1 and L_2 are two such Lagrangians, then it is easy to see that, when computing $\hom_{\mathcal{F}}(\Delta^{\gamma}_{L_1}, \Delta^{\gamma}_{L_2})$, one can make a choice of positive perturbation ϕ_ϵ such that the resulting intersection points exactly coincide with $L_0 \cap L_1$ (if that intersection was tranverse). Continuing that observation, we have:

Proposition 17 (Compare [AAK], Cor 7.7)**.** *If the normal bundle of M^{crit} is oriented and Spin, the thimble construction gives a full and faithful embedding*

$$\Delta^{\gamma}_{(\cdot)} : \mathcal{F}(M^{crit}) \hookrightarrow \mathcal{F}(E, W) \tag{1.13}$$

For a discussion of the sign problems that occur for non-Spin normal bundles, see [AAK].

Remark 18. As with E, we will assume M^{crit} is monotone or exact, or ... @@INSERT, hence has a well-defined Fukaya category.

Definition 19. *We say M^{crit} is* non-degenerate *if it admits a collection of Lagrangians $\{L\}$ that split-generate its Fukaya category, in the sense of* [A]*; that is if the map from the Hochschild homology of these Lagrangians to quantum cohomology of M^{crit} hits 1.*

We show that thimbles over any such collection of Lagrangians in M^{crit} hit 1 in $HF^*(E, W)$, which implies by Theorem 14 and Proposition 17 that:

Theorem 20. *If M^{crit} is non-degenerate, then the thimble functor is a derived equivalence.*

Remark 21. There are many known examples of *non-degenerate* M^{crit} (in the case M is monotone): toric Fanos, Fano hypersurfaces in projective space, various other homogeneous spaces, etc. @@CITE.

1.4.2. *Stabilizing the Fukaya category.* Kontsevich introduced in [K3] a notion of (dimensional) *stabilization* for a symplectic manifold, where the result of stabilization is no longer just a symplectic manifold but an LG model:

$$\Sigma M := (M \times \mathbb{C}, W = z^2),$$

where z denotes the coordinate on $\mathbb{C}$; and proposed that the stabilized manifold should have an identical pseudoholomorphic curve theory (or at least, an identical Fukaya category). The LG model ΣM is a Morse-Bott fibration with one critical component, M which has trivial normal bundle (in particular, it is tautologically oriented and Spin). Hence, an immediate corollary of Theorem 20 proves Kontsevich's proposal for manifolds M whose Fukaya categories are non-degenerate:

Corollary 22. *Let $\Sigma^k M := (M \times \mathbb{C}^k, W = \sum_{i=1}^k z_i^2)$. Then whenever M is non-degenerate in the sense of Definition 19, the full and faithful embedding given by the thimble functor is a Morita equivalence:*

$$\mathrm{perf}(\mathcal{F}(M)) \xrightarrow{\sim} \mathrm{perf}(\mathcal{F}(M \times \mathbb{C}^k, W = \sum_{i=1}^k z_i^2)). \tag{1.14}$$

2. The Floer homology of a Landau-Ginzburg model

2.1. **Landau-Ginzburg models.** Let (M, ω) be a compact symplectic manifold with boundary, and let μ be a symplectomorphism. The (one-ended) *symplectic mapping torus* of (M, μ) is defined to be the symplectic manifold

$$E_{(M,\mu)} \equiv \frac{M \times [0, 2\pi] \times [1, \infty)}{(x, 0, r) \sim (\mu(x), 2\pi, r)} \tag{2.1}$$

with symplectic structure $\omega_M + r dr \wedge d\theta$. Using polar coordinates, we obtain a projection

$$E_{(M,\mu)} \to \mathbb{C} \tag{2.2}$$

whose image is the complement of the unit disc.

Let (E, ω_E) be a symplectic manifold, and $W\colon\ E \to \mathbb{C}$ a proper map. Let $E^r = W^{-1}(D_r^2)$. We say that W is *modeled after a mapping torus* if there exists a symplectic mapping torus of a pair (M, μ) where M and a symplectomorphism

$$E_{(M,\mu)} \to E \setminus E^{in} \equiv E^{out}. \tag{2.3}$$

which is compatible with projection to $\mathbb{C}$. Such a model, if it exists, is unique up to symplectomorphism: M is the fibre of E at 1, and μ is the monodromy around the unit circle. The commutativity of the above diagram implies that the embedding $E_{(M,\mu)} \to E$ is given by parallel transport. Moreover, to simplify the discussion of Floer theory, we assume

(2.4) all fixed points of μ are non-degenerate and contained in the interior of M.

Remark 2.1. Starting with a compactly supported symplectomorphism, one can easily produce a symplectomorphism as above by composition with the Hamiltonian flow of a function which vanishes at the boundary with non-zero derivative.

We shall need some additional assumptions in order to be able to define a Fukaya category for the potential W, and define a Floer cohomology group which will play the role of the quantum cohomology of a closed symplectic manifold. To this end, we impose the following standard monotonicity/exactness assumptions (see [S2] in the exact situation, and Seidel genus 2, Sheridan Fano):

(1) M is either exact or positively monotone: we fix a primitive λ for ω in the first case, and for the pullback of ω to the unit circle bundle SM of the top (complex) exterior power of TM, which we call the anti-canonical bundle.
(2) μ is exact: the difference $\mu^*\lambda - \lambda$ is exact. If M is monotone, we use the fact that the action of μ lifts canonically to the anti-canonical bundle. We pick a function N on $M \times [0, 2\pi]$ in the exact case and $SM \times [0, 2\pi]$ in the monotone case, vanishing to infinite order at 0 and which agrees to infinite order at 2π with $\mu^*\lambda - \lambda$.
(3) E is either exact or positively monotone, and there is a choice of primitive whose restriction to E^{out} agrees with

$$\lambda_M + \frac{r^2}{2} d\theta + dN. \tag{2.5}$$

We also need assumptions which imply compactness for moduli spaces of holomorphic curves

(1) A semi-convex structure (ρ_M, J_M) on M: consisting of a function $\rho_M \colon M \to [-1, 0]$ with non-vanishing differential on ∂M such that $\rho_M^{-1}(0) = \partial M$, and a compatible almost complex structure J_M such that

$$dd^c \rho_M \geq 0 \tag{2.6}$$

$$d\rho_M \neq 0. \tag{2.7}$$

(2) μ preserves ρ_M and J_M near the boundary.
(3) A semi-convex structure (ρ_E, J_E) on E restricting to (ρ_M, J_M) on each fibre of W over E^{out}.

Given a semi-convex structure on E, we obtain a space $\mathcal{J}$ of ω_E-tame almost complex structures which agree with J_E near ∂E and such that map $W \colon E \to \mathbb{C}$ is holomorphic outside a compact set ($\mathbb{C}$ is equipped with the standard complex structure).

2.2. **Boundary rotation of the disc.** We say that a Hamiltonian $H \colon \mathbb{C} \times S^1 \to \mathbb{R}$ is *radial* (at infinity) if

$$H_t(re^{i\theta}, t) = f_t(\theta) r^2 \tag{2.8}$$

outside a compact set, and H vanishes on the unit disc. As before, $\mathbb{C}$ is equipped with the symplectic form $rdr \wedge d\theta$. A symplectomorphism $\phi \colon \mathbb{C} \to \mathbb{C}$ is *radial* (at infinity) if it is generated by a radial Hamiltonian; we denote the class of such symplectomorphisms by $\mathcal{S}$. A subset of $\mathbb{C}$ is *radial* (at infinity) if it is preserved by (real) dilation outside a compact set.

Lemma 2.2. *The image of a radial subset under a aadial symplectomorphism is radial.* $\square$

A radial Hamiltonian *generates a boundary rotation of angle* b if it agrees with $br^2/2$ outside a compact subset (in particular, it is autonomous). Let $\mathcal{S}_b(\mathbb{C})$ denote the Hamiltonian flow generated by such function. Given an interval $I \subset S^1$, and a constant ϵ, we say that

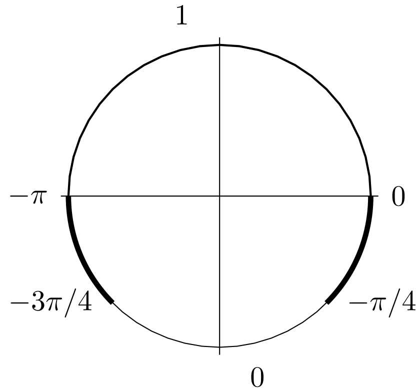


Figure 3. The function on the circle

the restriction of a radial Hamiltonian to the sector $\mathbb{C}_I \equiv \{re^{i\theta} | \theta \in I\}$ is an *ϵ-approximate boundary rotation of angle b* if

$$\left| \int_{t \in S^1} 2f_t(\theta + t)dt - b \right| \leq \epsilon \text{ for all } \theta \in I. \tag{2.9}$$

Remark 2.3. The angular term in the Hamiltonian vector field of H_t is $2f_t(\theta)\partial_\theta$. The above equation therefore implies that the image of any point of the plane with phase $\theta \in I$ has image of phase within ϵ of the section $I + b \subset S^1$.

we write $\mathcal{S}_b^r(\mathbb{C})$ for those which are given by rotation outside the disc of radius r.

Consider the class $\mathcal{X}$ of functions on the circle $\mathbb{R}/2\pi\mathbb{Z}$ whose derivative with respect to θ is non-negative for $\theta \in [-\pi/4, 0]$, non-positive for $\theta \in [-\pi, -3\pi/4]$, and which respectively agree with 1 and 0 on the intervals $[0, \pi]$ and $[-3\pi/4, -\pi/4]$. The regions where such functions are constant, monotonically increasing, or decreasing are shown in Figure 3.

Let $\mathcal{S}_\sigma(\mathbb{C})$ denote the space of the Hamiltonian flows of functions of the form $h \cdot \chi_r(\theta)$, where h generates a boundary rotation of angle $\pi + 2\epsilon$, and satisfies $h''(r) > 0$ whenever $h'(r) = \pi$, and χ_r is a family of functions in $\mathcal{X}$ such that

$$\frac{d\chi_r(\theta)}{dr} \geq 0 \tag{2.10}$$

$$\chi_r(\theta) = 1 \text{ if } h'(r) = \pi + 2\epsilon \text{ and } \theta \in [-\epsilon, 0] \tag{2.11}$$

; we write $\mathcal{S}_\sigma(\mathbb{C}, D_r)$ for those whose flows are constant in the disc of radius r. Figure 4 illustrates the restriction to a circle of large radius of a Hamiltonian diffeomorphism in $\mathcal{S}_\sigma(\mathbb{C})$. Note that each element $\phi_\sigma \in \mathcal{S}_\sigma(\mathbb{C})$ is eventually radially invariant in the sense that

$$r \cdot \phi_\sigma(z) = \phi_\sigma(r \cdot z), \tag{2.12}$$

whenever $|z|$ is sufficiently large and r is greater than 1. This can be alternatively restated in terms of the the graph $\Delta_\sigma \subset \mathbb{C}^2$ consisting of pairs $(z, \phi_\sigma z)$. Outside a compact set, this is invariant under rescaling the two variables.

Lemma 2.4. *Every element of $\mathcal{S}_\sigma(\mathbb{C}, D_r)$ agrees with identity in the sector $\theta \in [-\pi + 4\epsilon, -2\epsilon]$, and with rotation by $\pi + 2\epsilon$ along the ray $\theta = 0$ outside the disc of radius r. Moreover, there is a unique point outside the unit disc which is mapped to an antipodal point (on the circle of the same radius).*

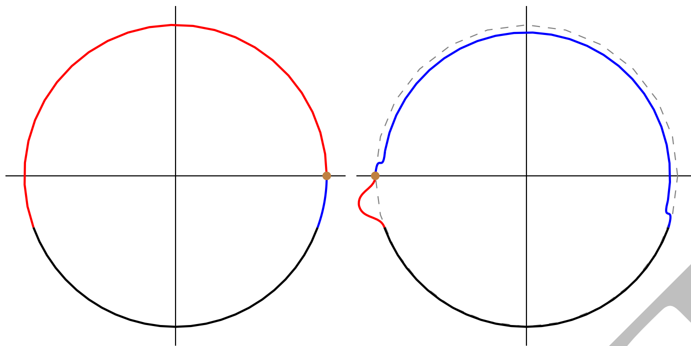


FIGURE 4. The left picture is the circle of radius satisfyin $h'(r) = \pi$. The right picture is its image under the Hamiltonian flow of $\chi(\theta) \cdot \pi r$. The black region is mapped identically to itself. The brown point is rotated by π, and the images of the red and blue regions are as shown.

Proof. The first part is obvious. To prove the second part, let $h(r)$ denote the function generating such a flow. It is clear from Figure 4 that the point $\theta = 0$ satisfying $h'(r) = \pi$ is mapped to an antipodal point. Such a point is unique because we assumed that $h''(r)$ is strictly positive at such points, and h is monotone. To prove that no other solution exists, we consider the following cases:

Case 1 $h'(r) > \pi$: Since $\chi(\theta) \leq 1$, $\phi(r, \theta)$ has argument strictly less than $\theta + \pi$.

Case 2 $h'(r) = \pi$: This is the case illustrated in Figure 4. Whenever $\theta \in (0, \pi + \epsilon)$, the image lies in a circle of larger radius, while if $\theta \in (-\epsilon, 0)$ the image lies in a circle of smaller radius. All points in the remaining interval are fixed.

Case 2 $h'(r) > \pi$: Again, whenever $\theta \in (0, \pi + \epsilon)$, the image lies in a circle of larger radius. On the other hand, if $\theta \in (-\epsilon, 0]$ then either the point lies in a circle of smaller radius, or it is rotated by more than π.

□

Let $\mathcal{S}_\sharp(\mathbb{C})$ denote the space of Hamiltonian diffeomorphisms which are composition $\phi_\sigma \circ \phi_\pi$ of a boundary rotation in $\mathcal{S}^r_\pi(\mathbb{C})$ and an element of $\mathcal{S}_\sigma(\mathbb{C}, D_r)$. The following result follows easily from the definitions and Lemma 2.4:

Proposition 2.5. *For each element $\phi_\sharp \in \mathcal{S}_\sharp(\mathbb{C})$ there are boundary rotations ϕ_π and $\phi_{2\pi+2\epsilon}$ such that the restriction of $\phi_\sharp$ agrees with*

(1) *ϕ_π in the sector $\theta \in [-\pi + 4\epsilon, -2\epsilon]$ and in the disc of radius r.*
(2) *$\phi_{2\pi+2\epsilon}$ in the sector $\theta \in [-\epsilon, \epsilon]$*

Moreover, $\phi_\sharp$ has exactly one fixed point outside the disc or radius 1 (which is non-degenerate), and is radially invariant outside a compact set.

It is convenient to introduce a class of (area preserving) diffeomorphisms of the plane which include $\mathcal{S}_\sharp(\mathbb{C})$, $\mathcal{S}_\sigma(\mathbb{C})$, and $\{\mathcal{S}^r_b(\mathbb{C})\}$: let $\mathcal{S}(\mathbb{C})$ denote the set of symplectomorphisms which are *radial outside a compact set* in the sense that

$$r \cdot \phi(z) = \phi(r \cdot z) \tag{2.13}$$

whenever z lies outside a compact subset and $r > 1$.

2.3. **Dualising symplectomorphisms.** In order to lift the discussion of the previous section to the total space E of a Landau-Ginzburg model, we begin by fixing a constant ϵ greater than 0, so that the Hamiltonian flow of ρ_E admits no closed orbits of time less than ϵ near ∂E. The existence of such a constant follows from the fact that E is modelled after a mapping cylinder outside a compact set.

If $1 < r$, let $\mathcal{S}_\sigma(W, E^r)$ denote the space of Hamiltonian diffeomorphisms which are time-1 flows of Hamiltonians of the form:

$$h_\sigma \circ W + H^\epsilon_\rho, \tag{2.14}$$

where h_σ generates an element of $\mathcal{S}_\sigma(\mathbb{C}, D_r)$, and H^ϵ_ρ is a time-dependent Hamiltonian E which, outside a compact subset, agrees with $\epsilon\rho_E$. Note that the condition $1 < r$ implies that the Hamiltonian vector field of $h_\sigma \circ W$ is parallel to the boundary since it vanishes in E^{in} and is pulled back from the cylinder on the mapping torus. Similarly, given a real number b, the space of boundary rotation of E of angle b, denoted $\mathcal{S}_b(W, E_r)$ is the space of Hamiltonian diffeomorphisms which are time-1 flows of Hamiltonians of the form:

$$h_b \circ W + H^\epsilon_\rho, \tag{2.15}$$

where h_b generates a boundary rotation in $\mathcal{S}^r_b(\mathbb{C})$ (i.e. agrees with $br^2/2$ outside a compact set).

Definition 2.6. *The space of* dualising symplectomorphisms $\mathcal{S}_\sharp(W)$ *is the space of Hamiltonian diffeomorphisms of* E *which agree with a composition* $\phi_\sigma \circ \phi_\pi$ *of elements of* $\mathcal{S}_\sigma(W, E^r)$ *and* $\mathcal{S}_\pi(W, E^r)$.

It is convenient to introduce a set $\mathcal{S}(W)$ of *radial symplectomorphisms* which are those symplectomorphisms ϕ generated by Hamiltonians of the form

$$h \circ W + H^\epsilon_\rho, \tag{2.16}$$

where h generates a radial symplectomorphism $\phi_\mathbb{C}$ of $\mathbb{C}$, and H^ϵ_ρ is as in Equations (2.14) and (2.15). In this case, we write that ϕ is a *lift* of $\phi_\mathbb{C}$. Because the second term in Equation (2.16) can be chosen arbitrarily in a compact subset of the interior of E, the correspondence between elements of $\mathcal{S}_\sharp(W)$ and their lifts is not unique (nor is the inverse correspondence uniquely defined). However, we have

$$W(\phi(x)) = \phi_\mathbb{C}(W(x)), \tag{2.17}$$

whenever x lies outside a compact subset of E. In particular, whenever $\phi_\mathbb{C}$ has no fixed points outside a compact set, the fixed points of ϕ lie in a compact subset.

Lemma 2.7. *There is a dense subset of* $\mathcal{S}(W)$ *consisting of symplectomorphisms with only finitely many fixed points, all of which are moreover non-degenerate and lie in the interior of* E. □

Proof. The subset of $\mathcal{S}(\mathbb{C})$ consisting of diffeomorphisms with no fixed points outside a compact set is dense, so it suffices to consider lifts of such a map $\phi_\mathbb{C}$. To show that the compact set in E containing all fixed points can be chosen to lie in the interior, observe that fixed points near ∂E give rise to time-ϵ Hamiltonian orbits of ρ_E; ϵ was chosen so that there are no such orbits. Since the second term in Equation (2.16) can be chosen arbitrarily in compact subsets of the interior of E, the result follows. □

2.4. **Graph correspondences.** Consider the symplectic manifold $E^- \times E$, which is the product of two copies of E equipped with the symplectic form $(-\omega, \omega)$. Given a symplectomorphism ϕ of E, we define the *graph* to be

$$\Gamma_\phi := \{(x, \phi x) \in E^- \times E\}. \tag{2.18}$$

The graph of the identity map will be denoted Δ_E. and the fixed points of ϕ are in bijective correspondence with the intersections of Γ_ϕ with Δ_E. We shall presently use this construction to define the Floer cohomology of symplectomorphisms as a Lagrangian Floer cohomology group, but we begin with some geometric preliminaries in this section.

In the basic case of $\mathbb{C}^- \times \mathbb{C}$, the condition that a symplectomorphism of $\mathbb{C}$ lie in $\mathcal{S}(\mathbb{C})$ is equivalent the fact that its graph is (outside a compact set) invariant under dilation in $\mathbb{C}^- \times \mathbb{C}$. In the general case, we note that, if $\phi \in \mathcal{S}(W)$ lifts $\phi \in \mathcal{S}(\mathbb{C})$ then

$$(W, W)\colon\ E^- \times E \to \mathbb{C}^- \times \mathbb{C} \tag{2.19}$$

maps Γ_ϕ to Γ_ϕ.

2.5. **Compactness for holomorphic strips.** Let (ϕ, ϕ') be a pair of radial symplectomorphisms of E which are generic in the sense of Lemma 2.7, i.e. so that all intersection points between them are transverse and contained in a compact set of the interior of E. Given an almost complex structure $J \in \mathcal{J}$ (see the end of Section 2.1), we define $\mathcal{M}(x, y)$ to be the moduli space of J-holomorphic maps from $\mathbb{R} \times [0, 1]$ to $E^- \times E$, mapping $\mathbb{R} \times \{0\}$ to Γ_ϕ and $\mathbb{R} \times \{1\}$ to $\Gamma_{\phi'}$, and converging to y at the positive end and x at the negative end. We let $\overline{\mathcal{M}}(x, y)$ denote the Gromov-Floer bordification (i.e. obtained by allowing all stable maps), and $\overline{\mathcal{M}}_k(x, y)$ the component consisting of discs of Maslov index k.

Lemma 2.8. *$\overline{\mathcal{M}}_k(x, y)$ is compact.*

Proof. It suffices to prove that there is a compact subset of the interior of E which contains the image of curves in $\overline{\mathcal{M}}(x', y')$ for all pairs (x', y') of intersections. Consider the projection to $\mathbb{C}^- \times \mathbb{C}$. This is a $(-j.j)$-holomorphic curve, with boundary mapping to two Lagrangians which are radially invariant (and disjoint) outside a compact set. The composition of this curve with the distance to the origin therefore satisfies Neumann boundary conditions, so that the maximum principle implies that all solutions are contained in a compact set (i.e. a fixed ball in whose complement the Lagrangians are disjoint and radially invariant). To conclude compactness, we apply the same maximum principle to the composition with ρ. □

For generic choices of $J \in \mathcal{J}$, $\overline{\mathcal{M}}_k(x, y)$ is regular if $k < 2$; in particular, this moduli space is empty if $k < 0$, consists of finitely many points if $k = 0$, and of a finite unions of circles and intervals if $k = 1$.

2.6. **Floer cohomology of graphs.** Given a class $\beta \in H^2(E, \mathbb{Z}_2)$ which we call a *background class*, we shall now define the *β-twisted Floer cohomology groups* associated to (generic) pairs of radial symplectomorphisms. We shall construct the groups using Lagrangian Floer cohomology in $E^- \times E$.

We begin by considering the class

$$(\beta + w_2E, \beta) \in H^2(E^-, \mathbb{Z}_2) \oplus H^2(E, \mathbb{Z}_2) \subset H^2(E^- \times E, \mathbb{Z}_2). \tag{2.20}$$

The diagonal $\Delta_E \subset E$ has a canonical Spin structure relative to this class in the sense of Fukaya-Oh-Ohta-Ono [???], i.e. representing $(\beta + w_2E, \beta)$ by an orientable vector bundle on the 2-skeleton of $E^{-} \times E$, we have a Spin structure on the direct sum of the restriction of this vector bundle to the diagonal with the tangent space TE. Indeed, we may choose such a vector bundle to be $(TE \oplus V) \boxplus V$, so that the restriction to the diagonal agrees with $V^{\oplus 2} \oplus TE$. Taking the direct sum with the tangent space yields $(V \oplus TW)^{\oplus 2}$, which has a canonical Spin structure since E since it is a direct sum of two orientable vector bundles.

Given $\phi \in \mathcal{S}(W)$ is a radial symplectomorphism, we equip Γ_ϕ with the Spin structure induced from the homotopy to the diagonal, using the fact that ϕ is homotopic to the identity. Assume that (ϕ, ϕ') is a transverse pair of Lagrangians, we can then assign to each transverse intersection point $x \in \Gamma_\phi \cap \Gamma_{\phi'}$ a $\mathbb{Z}_2$-graded free abelian group o_x, following the standard Floer-theoretic framework [?Seidel??]. The grading arises from the sign of the intersection of Γ_ϕ and $\Gamma_{\phi'}$, which are naturally oriented: in dimension n, the degree of o_x is even if the intersection number is $(-1)^n$, and odd otherwise.

We then define

$$CF^*(\phi, \phi'; \beta) = \bigoplus_{\substack{x \in \Gamma_\phi \cap \Gamma_{\phi'} \\ |\mathrm{o}_x| = *}} \mathrm{o}_x. \tag{2.21}$$

The Spin structures on the Lagrangian boundary conditions relative to the class $(\beta + w_2 E, \beta)$ yields maps

$$\mathrm{o}_y \to \mathrm{o}_x \tag{2.22}$$

wist by de-
ee?

associated to every element $u \in \overline{\mathcal{M}}_0(x, y)$ defines a canonical map . The sum of all such maps defines the differential

$$\partial \colon CF^{*-1}(\phi, \phi'; \beta) \to CF^{*-1}(\phi, \phi'; \beta). \tag{2.23}$$

The Floer cohomology will be denoted $HF^*(\phi, \phi'; \beta)$.

2.7. **Floer cohomology groups associated to Landau-Ginzburg models.** We now consider some Floer cohomology groups associated to the distinguished classes of radial symplectomorphisms introduced in Section 2.3.

Lemma 2.9. *If $b \neq 2\pi k$, the Floer cohomology $HF^*(\mathrm{id}, \phi; \beta)$ is independent of the choices of elements $\phi \in \mathcal{S}_b(E)$.*

dd reference
P's book

Proof. By construction, any pair of elements of $\mathcal{S}_b(E)$ agree outside a compact set, and assuming that $b \neq 2\pi k$ implies that such graphs are disjoint from the diagonal of E outside a compact set. Continuation maps with respect to compactly supported isotopies therefore yield the desired isomorphism. □

We shall therefore denote these group by $HF^*_b(W; \beta)$. To completely justify this notation, we need to address the choice of constant ϵ fixed at the beginning of Section 2.3.

Lemma 2.10. *The group $HF^*_b(W; \beta)$ is independent of the choice of constant ϵ, for which ρ_E admits no closed orbits of time less than ϵ.*

Proof. Let ϵ_t be a path of such constants. Given $\phi \in \mathcal{S}_b(E)$, defined with respect to ϵ_0, to suffices to construct a path ϕ_t of symplectomorphism, all of which agree with ϕ_0 in a compact set where all intersection points occur. The isomorphism can then induced by a canonical identification of Floer complexes or, equivalently, by the homotopy method isomorphism.

We define the family ϕ_t as $\psi_t \circ \phi$, where ψ_t is the Hamiltonian flow of a function $f_t(\rho_E) \cdot \rho_E$, such that $f_t \geq 0$, and $f_t(0) = \epsilon_t - \epsilon_0$. $\square$

We now consider $\phi \in \mathcal{S}_\sharp(E)$; the argument of Lemma 2.9 again shows that the Floer cohomology $HF^*(\mathrm{id}, \phi; \beta)$ does not depend on the choice of such an element, and the argument of Lemma 2.10 that it does not depend on ϵ. We denote this group by $HF^*(W; \beta)$.

Lemma 2.11. *If $b < 2\pi$ there is a canonical homomorphism $HF^*_b(W; \beta) \to HF^*(W; \beta)$.*

Proof. We begin by choosing a generic element $\phi^{\mathbb{C}} \in \mathcal{S}_\sharp(\mathbb{C})$ so that all fixed points in the annulus of biradius $(1, r)$ are non-degenerate. We fix a lift ϕ of this element to $\mathcal{S}_\sharp(\mathbb{C})$ so that Γ_ϕ projects to Γ_{phi^C} outside the product of two discs of radius 1; this can be achieved by assuming that the second term in Equation (2.16) agrees with a multiple of ρ outside E^1.

The fixed point of ϕ By Assumption (2.4)

By Proposition Recall that $\mathcal{S}_b(E)$ After possibly cutting off the function ρ_M, we may assume that the $\square$

9/2015 TODO:

(1) If H^ϵ_ρ is generic, then the Floer cohomology of (Δ, Δ_b) and $(\Delta, \Delta_\sharp$ is well defined with respect to this auxiliary Hamiltonian.
(2) These groups are independent of sufficiently small ϵ (homotopy method? continuation maps seem ok, but only go in one direction. Use a careful choice of Hamiltonian so that the continuation map is the identity. Then, modify Hamiltonian in compact set). and choice of Hamiltonian H^ϵ_ρ (continuation maps).
(3) Point out there is a map $HF^*(\Delta, \Delta_b) \to HF^*(\Delta, \Delta_\sharp)$, and in particular, a canonical element $1 \in HF^*(\Delta, \Delta_\sharp)$.
(4) Note that $HF^*(\Delta, \Delta_\sharp)$ is a special group and has a special name, $HF^*(W)$.
(5) Make a short remark that $HF^*(\Delta, \Delta_b) = HF^*(\phi_b)$ (potentially bijection of chain complexes), and the definition of $HF^*(W)$ is designed to avoid difficulties in working directly with $HF^*(\phi_\sharp)$.

Potentially deprecated:

$$\phi^\lambda_b \equiv \phi_b \circ \phi^\lambda_v \tag{2.24}$$

$$\phi^\lambda_\sharp \equiv \phi^t_\sharp \circ \phi^\lambda_v. \tag{2.25}$$

Since the Hamiltonian flow of ρ commutes with the flow of functions pulled back from the base (which vanish on the disc), the order of compositionabove does not matter.

Consider the associated Lagrangian graphs

$$\Delta^\lambda_b \equiv \{(x, \phi^\lambda_b(x))\} \subset E^- \times E \tag{2.26}$$

$$\Delta^\lambda_\sharp \equiv \{(x, \phi^\lambda_\sharp(x))\} \subset E^- \times E. \tag{2.27}$$

Lemma 2.12. *The correspondences Δ^λ_b and $\Delta^\lambda_\sharp$ are admissible. Moreover, if K, L, and L' are admissible Lagrangians with A_L and $A_{L'}$ subsets of $(-\epsilon, \epsilon)$ and $A_K \subset (-\pi + 4\epsilon, -2\epsilon)$, the collection $(\Delta, \Delta^\lambda_\sharp, L \times L', K \times K, L \times K, K \times L')$ is admissible for $\lambda \notin \mathbb{Z}$. Moreover the following collections are isotopic through admissible:* This is still a mess

(1) $(\Delta, \Delta^\lambda_b, K \times K)$ *for $\lambda \notin \mathbb{Z}$ and $b = \pi$.*
(2) $(\Delta, \Delta^\lambda_\sharp, L \times L')$

3. The Fukaya category of a LG model

3.1. Admissible Lagrangians.

Definition 3.1. *An* admissible *Lagrangian submanifold $L \subset E$ is a submanifold with Legendrian boundary $\partial L \subset \partial E$ satisfying the following properties:*

- *L is properly embedded in E and disjoint from its boundary.*
- *Outside the unit disc, $\pi(L)$ agrees with a finite collection of arcs which are asymptotic to radial rays disjoint from the negative real axis.*
- *The restriction of $d\rho \circ J|L$ vanishes in a neighbourhod of ∂L.*

In order to avoid virtual fundamental chains, we require that L be either exact or monotone. Alternatively, if the Maslov index vanishes on $\pi_2(M, L)$, we may assume that the moduli space of J-holomorphic discs with one marked point with boundary L has image contained in a union of manifolds of codimension strictly greater than 2.

By considering the asymptotic directions of the rays which agree with $\pi(L)$ outside the unit disc, we can associate to each admissible Lagrangian a collection of angles

$$A_L \subset (-\pi, \pi) \tag{3.1}$$

We say that these angles are the *directions of L at infinity*. By definition, A_L is contained in an admissible sector of $(-\pi, \pi)$.

3.2. The Floer complex.

Fix a *background class* $b \in H^2(E, \mathbb{Z}_2)$, represented as the second Stiefel-Whitney class of a vector bundle V_b on E. A *relative* Pin *structure* on a Lagrangian L is the choice of a Pin structure on $TL \oplus V_b|L$. A admissible Lagrangian equipped with such a relative Pin structure is called an *admissible Lagrangian brane*.

Remark 3.2. The results of this paper hold over arbitrary rings, in particular over rings of characteristic 2. In this case, we need not require the existence (and hence not the choice) of a relative Pin structure in order for Floer theory to be well-defined.

Let L_0 and L_1 be admissible Lagrangian branes whose boundaries do not intersect, and such that

$$A_{L_1} \neq A_{L_0}. \tag{3.2}$$

We fix a generic pair $(J_{L_0,L_1}, H_{L_0,L_1}) \in \mathcal{J} \times \mathcal{H}_0$, and obtain a Floer complex

$$CF^*(L_0, L_1) = \bigoplus_{x \in \mathcal{X}(L_0,L_1)} \delta_x \tag{3.3}$$

where $\mathcal{X}(L_0, L_1)$ is the space of time-1 chords of H_{L_0,L_1} starting on L_0 and ending on L_1, and δ_x is the orientation line of such a chord, see [?seidel-book]. The degree $\deg(x) \in \{0, 1\}$ is ??

Given a pair (x_-, x_+) of such chords, the differential is obtained from the moduli space of holomorphic strips

$$\begin{aligned} \mathcal{R}(x_-, x_+) = \{u \colon \mathbb{R} \times [0,1] \to E | u(0,s) \in L_0, u(1,s) \in L_1, \\ \lim_{s \to \pm\infty} u(s,t) = x_\pm(t), (du - X_{L_0,L_1})^{0,1} = 0\} / \sim, \end{aligned} \tag{3.4}$$

where $\sim$ denotes taking the quotient by the $\mathbb{R}$ action. Let $\mathcal{R}^d(x_-, x_+)$ consist of those components of the moduli space consisting of element of virtual dimension d.

Lemma 3.3. *There is a compact subset of the interior of E containing the image of all elements of $\mathcal{R}(x_-, x_+)$*

9/15: no longer need the Neumann boundary maximum principle.

Proof. Apply the maximum principle to the projection to conclude that all elements of $\mathcal{R}(x_-, x_+)$ are contained in the inverse image of a compact subset in $\mathbb{C}$. Composing with ρ, and applying the maximum principle with Neumann boundary conditions implies that the image is in fact contained in a compact subset of E. □

The genericity assumption implies that those moduli spaces of virtual dimension less than or equal to 1 are regular. To each strip of virtual dimension 0 (rigid), we can canonically assign a map

$$\mu_u^1 \colon \delta_{x_+} \to \delta_{x_-}. \tag{3.5}$$

The differential is defined as the sum

$$\mu^1|\delta_{x_+} = \bigoplus_{x_-} \sum_{u \in \mathcal{R}^0(x_-, x_+)} \mu_u^1. \tag{3.6}$$

The finiteness of this sum is a consequence of Lemma 3.3.

3.3. A_∞ **operations.** Let $\overline{\mathcal{R}}_k$ denote the Stasheff associahedron, i.e. the moduli space of discs with $d+1$ boundary punctures one of which is distinguished as an outgoing points. Assume that consistent choices of strip-like ends $\{\epsilon_i\}_{i=0}^k$ along each end of a Riemann surface representing an element of $\overline{\mathcal{R}}_k$ has been fixed as in [?seidel-book][Section ??]; our convention will be that the end ϵ_0 is negative, while all other ends are positive. Let $\overline{\mathcal{U}}_k$ denote the universal curve over $\overline{\mathcal{R}}_k$, whose fibre as a point S is the Riemann surface S.

Let $\vec{L} = (L_0, \dots, L_k)$ be a sequence of admissible Lagrangian branes whose asymptotic ends are pairwise disjoint, which induce a labelling on the boundary components of every surface $S \in \overline{\mathcal{U}}_k$, starting counterclockwise at the outgoing puncture. Let $\partial_i S$ be the component labelled by L_i..

The definition of the A_∞ operation requires the choice of parametrised Floer data

$$J_{\vec{L}} \colon \overline{\mathcal{U}}_k \to \mathcal{J} \tag{3.7}$$

$$H_{\vec{L}} \colon \overline{\mathcal{U}}_k \to \mathcal{H}_0 \tag{3.8}$$

subject to the following constraints:

(1) The pull back of these data under the strip-like end corresponding to the pair (L_i, L_{i+1}) agree with the data $(J_{L_0,L_1}, H_{L_0,L_1})$.
(2) Near the boundary of $\overline{\mathcal{U}}_k$, these data are obtained by gluing.

A standard technique [?seidel-book][Section ??] yields the construction of data satisfying these properties by induction on k. Given a sequence $\vec{x} = (x_0, \dots, x_k)$ such that x_i is a chord of H_{L_{i-1},L_i}, we obtain a moduli space of holomorphic polygons:

(3.9)
$$\mathcal{R}_k(x_0, \dots, x_k) = \{u \colon S \to E | S \in \overline{\mathcal{R}}_k, u\partial_{L_i} S \subset L_i, \lim_{s \to \pm\infty} \epsilon_i \circ u(s,t) = x_i(t), (du - X_{\vec{L}})^{0,1} = 0\}.$$

Here, $X_{\vec{L}}$ is the Hamiltonian vector field of $H_{\vec{L}}$, and the anti-holomorphic part is taken with respect to $J_{\vec{L}}$. Again, we have a decomposition of this moduli space by components $\mathcal{R}_k^d(x_0, \dots, x_k)$ of virtual dimension d. For generic data, those components of virtual dimension less than or equal to 1 are regular. We can then canonically assign a map

$$\mu_u^k \colon \delta_{x_k} \otimes \dots \otimes \delta_{x_1} \to \delta_{x_0} \tag{3.10}$$

to every rigid element of the moduli space. The k^{th} higher product is then the sum

$$\mu^k|\delta_{x_k} \otimes \cdots \otimes \delta_{x_1} = \bigoplus_{x_0} \sum_{u \in \mathcal{R}^0_k(x_0,\ldots,x_k)} \mu^1_u. \tag{3.11}$$

The finiteness of this sum follows by a staightforward generalisation of Lemma 3.3. The proof that these operations satisfy the A_∞-relation is by now standard.

3.4. **An auxiliary directed category.** Following [AS], we define a directed category $\mathcal{O}_W$ with objects admissible Lagrangian branes, and morphisms

$$\mathcal{O}_W(L_0, L_1) = \begin{cases} CF^*(L_0, L_1) & \text{if } A_{L_1} < A_{L_0} \\ \mathbb{Z} & \text{if } L_0 = L_1 \\ 0 & \text{otherwise.} \end{cases} \tag{3.12}$$

We recall that $A_L \subset S^1$ is the set of angular directions corresponding to an admissible Lagrangian, and that $A_{L_1} < A_{L_0}$ means that all angles associated to L_1 are larger than those associated to A_{L_0}. Equality means that not just the ends are equal, but that $L_0 = L_1$ as objects of the Fukaya category. The construction of the previous section naturally descends to an A_∞ structure on $\mathcal{O}_W$, with the extra proviso that the generator $1 \in \mathbb{Z} = \mathcal{O}_W(L, L)$ acts a strict unit.

Remark 3.4. There is an alternate version of this construction in which $\mathcal{O}_W(L, L) := CF^*(L, L)$; compactness for moduli spaces involved in the A_∞ structure maps with consecutive copies of $\mathcal{O}_W(L, L)$ can be achieved by using a Morse/pearly model for Floer co-chains of L, which does not require any perturbation of L near ∞. Ultimately, the eventual Fukaya categories obtained in either way are equivalent.

evise remark

For any sub-interval $I \in (-\pi, \pi)$, we have a full subcategory $\mathcal{O}^I \subset \mathcal{O}$ consisting of objects whose ends are contained in the interval I. By abuse of notation, we write $\mathcal{O}^b$ for the subcategory of objects corresponding to a fixed (small) neighbourhood of $b \in (-\pi, \pi)$.

3.5. **Quasi-units.** The Fukaya category of W as defined in [AS] is obtained by a localization process from the auxiliary directed category $\mathcal{O}$ with respect to a class of morphisms $\mathcal{Z} \subset H^0(\mathcal{O})$. Roughly speaking, we want to identify a given Lagrangian L with all Lagrangians L' which differ by L by a positive angular bend which does not cross $-\pi$. We do this by finding canonical elements $[q] \in H^0 \hom_{\mathcal{O}}(L', L))$, which we call *quasi-units*, and inverting these elements.

Remark 3.5. For Lagrangians L with multiple ends (which were not considered in [AS]), there may not be a choice of bend L' such that $A_{L'} > A_L$. We introduce a procedure to remedy this, described below.

In light of Remark 3.5, we will define the class of quasi-units in three steps.

Definition 3.6. *A* continuation element *is any element $q \in HF^0(\phi_\epsilon L, L)$ induced by continuation, where ϕ_ϵ is a generic radial bend by a small (generic) quantity ϵ which does not take L past $-\pi$.*

Remark 3.7. Note that for a Lagrangian with multiple ends, it may not be the case that $A_{\phi_\epsilon L} > A_L$, hence continuation elements are not always morphisms in $\mathcal{O}$.

Definition 3.8. *A* pre-quasi-unit *is any element $q \in HF^0(L_0, L_1)$ obtained from a continuation element by a sequence of the following operations:*

(1) Continuation maps $HF^0(L_0, L_1) \to HF^0(\phi_\epsilon L_0, L_1)$*, where ϕ_ϵ is a generic radial bend by a (n also generic) $\epsilon > 0$ which does not bend L_0 past $-\pi$.*
(2) homotopy isomorphisms $HF^0(L_0, L_1) \to HF^0(L_0', L_1)$ *induced by bending some (but not necessarily all!) of the ends of L_0 counterclockwise (but not passing $-\pi$), in a manner creating no new intersection points with L_1.*

Remark 3.9. In the second case, the *homotopy method* @@REF for establishing invariance of Floer cohomology produces the desired isomorphism $HF^0(L_0, L_1) \to HF^0(L_0', L_1)$ from the family of Lagrangians L_0^t realizing the bend. A special case of this is when the bend L_0^t agrees with L_0 inside a sufficiently large set containing all intersection points with L_1. Then, since holomorphic discs all stay within the same compact region, there is simply an equality of chain complexes $CF^*(L_0, L_1) = CF^*(L_0', L_1)$.

Insert reference for homotopy method

Insert where these equations for continuation are, and that they come from a maximum principle.

Definition 3.10. *The* collection of quasi-units *is the set of all pre-quasi-units q which live in the morphism spaces of $H^0(\mathcal{O})$; $q \in H^0(\hom(L_0, L_1))$. Namely, they are any pre-quasi-units $q \in HF^0(L_0, L_1)$ such that $A_{L_0} > A_{L_1}$.*

Example 3.11. @@Insert example of a Lagrangian with a wide set of ends, and how one determines a quasi-unit here.

The following Lemma will help guarantee that there are 'enough' quasi-units to compute morphisms.

Lemma 3.12. *Given any admissible Lagrangian L, for any sufficiently small interval $I \subset (\pi - \max(A_L), \pi)$, there is a Lagrangian L' supported in the interval I and a quasi-unit $q \in H^0\mathcal{O}(L', L)$.*

Proof. This is an inductive argument, depending on the number of ends and their relative locations; suppose the ends are at angles $r_1, \dots, r_k$. First, one produces a continuation element $q \in HF^0(L_\epsilon, L)$ for any sufficiently small generic $\epsilon \ll \frac{1}{2}\min\{r_{i+1} - r_i\}$ (with the convention that $r_{k+1} = r_k + (\pi - r_k)/2$; this has ends at $\{r_i + \epsilon\}$, Then, one bends the ends of L_ϵ, producing a new $L_\epsilon^{(1)}$ with new ends are at $\{r_2 - \epsilon, \dots, r_k - \epsilon, r_k + \epsilon\}$; using a homotopy isomorphism, we obtain a new element $q^{(1)}$. Next, by continuation, we obtain an element $q^{(2)} \in HF^0(\phi_\epsilon L_\epsilon^{(1)}, L_1)$, where $L_\epsilon^{(2)} := \phi_\epsilon L_\epsilon^{(1)}$ now has ends at $r_2 + \epsilon, r_3 + \epsilon, \dots, r_k + \epsilon, r_k + 2\epsilon$. (in particular, two of the ends are now "past" the last end of L). We repeat this process inductively, iteratively (in two steps) moving the $k - r + 1$st end of $L^{(2r)}$ past the kth end of L to form $L^{(2r+2)}$. Eventually, this produces $L' := L^{(2k)}$ and a quasi-unit as desired. Moreover, this Lagrangian lives in a $k\epsilon$ interval immediately past $\max A_L$. By taking ϵ sufficiently small and further pushing this Lagrangian closer to 2π by continuation, the Lemma is satisfied. ☐

The methods really apply to the subcategory $\mathcal{O}_J$ associated to any interval J; hence

Corollary 3.13. *Given any admissible Lagrangian in $\mathcal{O}_J$, for any sufficiently small interval $I \subset (\max J - \max(A_L), \max J)$, there is a Lagrangian L' supported in the interval I and a quasi-unit $q \in H^0\mathcal{O}(L', L)$.* ☐

3.6. **The Fukaya category.** The *Fukaya category of the Landau-Ginzburg model* $\mathcal{F}(W)$ is the category obtained from $\mathcal{O}(W)$ by localization: one inverts all quasi-units

$$\mathcal{F}(W) := \mathcal{Z}^{-1}\mathcal{O}(W). \tag{3.13}$$

For shorthand, we'll call this $\mathcal{F}$ from here on. By construction, $\mathcal{F}$ is equipped with a canonical localization functor

$$j : \mathcal{O} \to \mathcal{F}, \tag{3.14}$$

which sends morphisms whose cohomology classes are in $\mathcal{Z}$ to quasi-isomorphisms, and moreover is the (quasi-)universal such j.

An explicit construction of $\mathcal{F}$ can be given via the A_∞ quotient construction of [`?Lyubashenko:2008aa`, `?Lyubashenko:2006aa`] (the dg version appears in [`?Drinfeld:2004aa`, `?Keller:1999aa`]). Namely, given the collection $\mathcal{Z}$ one can associate a sub-category $\mathcal{A}$ of $Tw\mathcal{O}$ consisting of all mapping cones of morphisms in $\mathcal{A}$. The quotient construction of *loc. cit.* produces a new A_∞ category $Tw\mathcal{O}/\mathcal{A}$ equipped with a functor

$$Tw\mathcal{O} \to Tw\mathcal{O}/\mathcal{A}$$

which is quasi-universal among functors sending the objects of $\mathcal{A}$ to zero objects. The localization $\mathcal{F} := \mathcal{Z}^{-1}\mathcal{O}$ is defined then to be the image

$$\mathcal{O} \hookrightarrow Tw\mathcal{O} \to Tw\mathcal{O}/\mathcal{A}.$$

The construction of quotients of [`?Lyubashenko:2008aa`, `?Lyubashenko:2006aa`, `?Drinfeld:2004aa`] leads to very explicit (if extremely large) models for the morphism spaces in $\mathcal{F}$ (see e.g., @@REF). It is convenient (and crucial for our purposes) to have a simpler recipe for computing cohomological morphism spaces: if the Lagrangians are in the correct order, one can simply take their Floer cohomology.

Lemma 3.14 ('correct position' Lemma)**.** *Suppose* (K, L) *is a pair of objects of* $\mathcal{O}$ *such that* $K > L$*. Then,* j *induces an isomorphism*

$$[j^1] : HF^*(K, L) = H^*(\hom_{\mathcal{O}}(K, L)) \xrightarrow{\sim} H^*(\hom_{\mathcal{F}}(K, L)). \tag{3.15}$$

Proof. The basic observation, which does not quite prove the Lemma, is that

(3.16) if $K > L$, then multiplication by any quasi-unit on $H^*(\hom(K, L)) = HF^*(K, L)$ gives an isomorphism, whether on the left to $HF^*(K', L)$ with $K' > K$ or on the right to $HF^*(K, L')$ with $L > L'$.

(this fact is an immediate application of, for instance the homotopy method. E.g., if there is a quasi-unit from K' to K then by the definition of quasi-units, there is a one-parameter family of Lagrangians K^t interpolating between K and K' with $K^{t_1} > K^{t_0}$ whenever $t_1 > t_0$. Moreoever, when $K > L$, up to a compactly supported Hamiltonian perturbation, the complexes $\hom_{\mathcal{O}}(K^t, L)$ can be chosen to have the same generators and differentials, so they are all isomorphic)....

ave to say mething ore

There are two ways of turning the above observation into a proof of Lemma 3.14.

(1) The first method uses the traditional tool of Gabriel-Zisman localization @@CITE: Following @@REF we say that a localization pair $(\mathcal{C}, \mathcal{Y})$ (where $\mathcal{Y}$ is a class of morphisms) admits a *calculus of fractions*

Lemma 3.12 and (3.16) in conjunction imply that

(2) The second method (which is essentially equivalent to that done in [AS]) begins by applying directly the universal property of localizations to a functor built out of K. As a baby step, if, for a given K, the Yoneda A_∞ functor

$$\begin{aligned} \mathcal{Y}^{\mathcal{O}}_K : \mathcal{O} &\to Ch \\ L &\mapsto \hom_{\mathcal{O}}(K, L) \end{aligned}$$

sends all elements of $\mathcal{Z}$ to cohomology isomorphisms, meaning that for every L and every quasi-unit $L' \xrightarrow{q} L$, multiplication by q induced an isomorphism $H^*(\hom_{\mathcal{O}} K, L')) \xrightarrow{\sim} H^*(\hom_{\mathcal{O}} K, L))$, then it would follow that up to $\mathcal{Y}^{\mathcal{O}}_K$...

Of course, the above hypotheses does not hold in $\mathcal{O}$. For instance, if $L' > K$, the above isomorphism is not guaranteed. To remedy this, we consider, instead of K, a sort of 'injective resolution' of objects. Namely: by iteratively applying Lemma 3.12, given a fixed K, we can iteratively construct an infinite sequence $\cdots K_m \to K_{m-1} \to \cdots \to K_0 = K$ where all of the morphisms are quasi-units, and the ... Complete thought.

□

Remark 3.15. Lemma 3.14 gives a general recipe for computing the (cohomological) morphisms in $\mathcal{F}$ for an arbitrary pair K, L. Namely, one can always find a K' differing by K by a quasi-unit in $\mathcal{O}$ (hence K' is isomorphic to K in $\mathcal{F}$) satisfying $K > L$. Then, Lemma 3.14 says for any such K'

$$H^*(\hom_{\mathcal{F}}(K, L)) = HF^*(K', L). \tag{3.17}$$

There are similar prescriptions for computing the cohomological composition, or even sequences of chain level compositions. For instance, if one wants to understand

$$[\mu^2] : H^*(\hom_{\mathcal{F}}(L_1, L_2)) \otimes H^*(\hom_{\mathcal{F}}(L_0, L_1)) \to H^*(\hom_{\mathcal{F}}(L_0, L_2)), \tag{3.18}$$

one simply finds L_0'', L_1', each differing from L_0 and L_1 by a quasi-unit in $\mathcal{O}$, satisfying $L_0'' > L_1' > L_2$, and then takes the usual $[\mu^2]$ by counting discs (in $\mathcal{O}$):

$$HF^*(L_1', L_2) \otimes HF^*(L_0'', L_1') \to HF^*(L_0'', L_2). \tag{3.19}$$

3.7. **Fukaya categories of sectors.** Denote by $\mathcal{F}_{big}$ the localization of $\mathcal{O}$ with respect to all quasi-units Z_β constructed above. For any subcategory $\mathcal{O}^I, \mathcal{O}^b \subset \mathcal{O}$, we can localize by all quasi-units which remain within I to obtain a category $\mathcal{F}^I$, $\mathcal{F}^b$ (where b once more by abuse of notation refers to a small neighborhood around b). The inclusion $i_0 : \mathcal{O}^0 \subset \mathcal{O}$ send quasi-units to quasi-units, and in particular induces a functor $\bar{i}_0 : \mathcal{F}^0 \to \mathcal{F}_{big}$ fitting into a commutative diagram

Insert proof of Lemma: the main point is that any Lagrangian, no matter how far out its ends are, has a representative with all ends very closely pinched together.

$$\begin{tikzcd} \mathcal{O}^0 \arrow[r, "i_0"] \arrow[d, "j"] & \mathcal{O} \arrow[d, "j_{big}"] \\ \mathcal{F} \arrow[r, "\bar{i}_0"] & \mathcal{F}_{big} \end{tikzcd} \tag{3.20}$$

Lemma 3.16. $\bar{i}_0$ *is a quasi-equivalence.*

Define the *Fukaya category of the fibration* π to be

$$\mathcal{F} := \mathcal{F}^0. \tag{3.21}$$

(it is quasi-equivalent to any other $\mathcal{F}^I$). We will discuss the homological algebra of localization more shortly, but essentially $\mathcal{F}$ (and $\mathcal{F}_{big}$) is the universal category, with ob $\mathcal{F} =$ ob $\mathcal{O}^0$ (ob $\mathcal{F}_{big} =$ ob $\mathcal{O}$), satisfying the following properties:

- There is a functor $j : \mathcal{O}^0 \to \mathcal{F}$ (resp. $j_{big} : \mathcal{O} \to \mathcal{F}_{big}$).
- Quasi-units are sent via $j_{(big)}$ to isomorphisms.

4. The dualizing symplectomorphism on the Fukaya category

4.1. **Rotation bimodules.** Conceptually, it seems easier to first describe the bimodules induced by rotation by an angle:

(1) Given a sector of the fibration I, and a 'rotation amount κ' that 'stays clear' of $-\pi$ (meaning that rotation by this amount does not intersect the forbidden angle), construct a bimodule $\mathcal{B}_\kappa$ over the sector category $\mathcal{O}_I$, as the graph bimodule of a functor. For general nonsense reasons, one gets corresponding bimodules over $\mathcal{F}_I$ and $\mathcal{F}$ by transfer.
(2) argue that as long as the new sector $\phi_\kappa I$ is $> I$, that the resulting bimodule is *local*, meaning its cohomology groups are identical to the cohomology groups of the localized bimodule.
(3) two key examples given:
    - sector 0 and $2\pi + 2\epsilon$ rotation, which we will call $\mathcal{B}_{2\pi}$? We need a good name for this bimodule. **Key Lemma: This bimodule is isomorphic (on homology) to the Serre inverse bimodule. Draw pictures explaining.**
    - sector $-\pi/2$ and π rotation. Lemma: the resulting bimodule is quasi-isomorphic to the diagonal bimodule.

4.2. **The dualizing bimodule.** For compatibility with the closed string Floer homology group, we associate a bimodule to the class of symplectomorphisms of the form $\phi_\sharp$.

It is easiest to do this via a (simple) form of quilts:

(1) Construct $\Delta_\sharp$ as a bimodule over $\mathcal{O}_I$, where I is either one of the two sectors (either the rotate by π sector or the 2π sector.
(2) In fact, it is helpful more generally to construct $\Delta_\sharp$ as a $\mathcal{O}_I - \mathcal{O}_J$ bimodule, where I, J are one of the two sectors we are using (but allowed to be different). The reason is that later, in the coproduct map, one of the output modules will not be a Yoneda module, but will be $\Delta_\sharp$ thought of as a $\mathcal{O}_{-\pi/2} - \mathcal{O}_0$ bimodule (or rather restricted to being a left module).

4.3. **Equivalence on sectors.** The key propositions here are:

(1) Around sector 0, the dualizing bimodule is quasi-isomorphic to the $2\pi + 2\epsilon$ rotation bimodule; in particular, the localized bimodule is the desired dualizing bimodule.
(2) Around sector $-pi/2$, the dualizing bimodule is quasi-isomorphic to the π $(+\epsilon)$ rotation bimodule; in particular, the localized bimodule is quasi-isomorphic to the diagonal bimodule.

    This latter point continues to be true when $\mathcal{O}^\sharp$ is thought of as a $\mathcal{O}_{-\pi/2} - \mathcal{O}_0$ bimodule, which is what we will need later. (or rather, we will need the special case of restricting to the induced left module $\mathcal{O}^\sharp(K, -)$).

5. Pre-localized geometric structures

5.1. From open to closed.

(1) Construct the main open-closed map
$$\mathrm{CC}_*(\mathcal{O}_0, \mathcal{O}^\sharp) \to CF^*(E, W).$$
This will boil down to a certain quilted count.

(2) Say a word or two that for general nonsense reasons, there is an induced map from the Hochschild complex of the 2π rotation bimodule.

5.2. The coproduct. Let K be an object in sector $-\pi/2$.

(1) Construct a geometric coproduct map of $\mathcal{O}_0$ bimodules ${}_0\mathcal{O}_0^\sharp \to \mathcal{O}^\sharp(K, -) \otimes_{\mathbb{K}} \mathcal{Y}_K^r$. (counting quilts where the interesting seam with boundary on $\Delta^\sharp$ goes to the left factor).

(2) On at least the cohomology level, prove the following key Lemma: under the equivalence sending ${}_0\mathcal{O}_0^\sharp$ to $\mathcal{O}_{2\pi+2\epsilon}$, and for specific choices of representative Lagrangians, the coproduct is equivalent to a map counting triangles with two outputs. In particular, it is Serre dual to the multiplication map
$$\mu^2 : HF^*(\phi^2_{Serre}K, \phi_{Serre}K) \otimes HF^*(\phi_{Serre}K, K) \to HF^*(\phi^2_{Serre}K, K).$$

5.3. From closed to open.

(1) For any K in sector $-\pi/2$, construct a map $CF^*(E, W) \to \mathcal{O}^\sharp(K, K)$. Argue that for general reasons, one gets a map to the π rotation bimodule of K, K.

5.4. The collapse map. For K in sector $-\pi/2$, we define a collapse map $\mathcal{Y}_K^r \otimes_{\mathcal{O}_0} \mathcal{O}^\sharp(K, -) \to \mathcal{O}^\sharp(K, K)$. It very nearly comes from the bimodule structure map for $\mathcal{O}^\sharp(K, -)$ but not quite, as the K in $\mathcal{Y}_K^r$ is not in sector 0.

(1) *Compatibility check*: Under the quasi-isomorphism between $\mathcal{O}^\sharp(K, -)$ and $\mathcal{O}^\pi(K, -)$, the collapse map $\mathcal{Y}_K^r \otimes_{\mathcal{O}_0} \mathcal{O}^\sharp(K, -) \to \mathcal{O}^\sharp(K, K)$ is sent to another naturally defined collapse map $\mathcal{Y}_K^r \otimes_{\mathcal{O}_0} \mathcal{O}^\pi(K, -) \to \mathcal{O}^\pi(K, K)$. This collapse map is just a μ operation (coming from $\mathcal{O}$); in particular, it counts holomorphic discs.

(2) Remark: given how close this collapse map is to a bimodule structure map, it may be possible to have a 'general nonsense' argument for the previous point. But given the current absence of that, there is a geometric argument.

5.5. The Cardy condition. Via an annulus argument (which involves some quilted seams), realize the previous four sections as the sides of the following Proposition:

Proposition 5.1. *There is commutative diagram*

$$\begin{tikzcd}
\mathrm{CC}_*(\mathcal{O}^0, \mathcal{O}_\sharp) \arrow[r] \arrow[d] & \mathcal{Y}_K^r \otimes_{\mathcal{O}^0} \mathcal{Y}_K^\sharp \arrow[d] \\
CF^*(\Delta, \Delta_\sharp) \arrow[r] & CF^*(K \times K, \Delta_\sharp)
\end{tikzcd} \tag{5.1}$$

6. Localizing the diagram

INSERT

7. Applications

7.1. **Lefschetz fibrations with one critical point.** Consider now the case of the model Lefschetz fibration $(E, W) = (\mathbb{C}^n, \sum_i z_i^2)$. We denote by K the standard thimble fibred over the arc $\mathbb{R}_+$, equal to the subspace $\mathbb{R}^n \subset \mathbb{C}^n$. Note that $\hom_{\mathcal{F}(W)}(K, K) = \mathbb{K}$ in degree zero, and in particular the A_∞ structure is completely trivial. In this case, it is easy to calculate the closed string Floer homology group:

Lemma 7.1. $H^*(\mathbb{C}^n, \sum_i z_i^2) = H^*(\mathbb{C}^n) = \mathbb{K}$.

Proof. The monodromy at ∞ is a standard Dehn twist and has no fixed points. $\square$

Because $[\mathcal{CO}_K] : H^*(E, W) \to H^*(\hom_{\mathcal{F}}(K, K))$ sends 1 to 1, we conclude

Corollary 7.2. *The closed-open map* $[\mathcal{CO}_K]$ *is an isomorphism.*

Using these facts along with general facts about the monodromy $\phi_{2\pi}$, we can conclude

Proposition 7.3. *K satisfies the generation criterion, i.e., the hypotheses of Theorem @@REF.*

Proof. There is a natural map $[\mathcal{OC}_0] : HF^*(\phi_{2\pi}K, K) \to HF^*(E, W)$ which is the (cohomology of the) zeroth order term of $\mathcal{OC} : CC_*(\mathcal{O}, \mathcal{O}^\sharp) \to HF^*(E, W)$, and in particular factors as $HF^*(\phi_{2\pi}K, K) \to \mathrm{HH}_*(\mathcal{O}, \mathcal{O}^\sharp) \to HF^*(E, W)$. It suffices to prove that $[\mathcal{OC}_0]$ is an isomorphism.

But by Corollary 7.2, it suffices to verify that $[\mathcal{CO}_K] \circ [\mathcal{OC}_0] : HF^*(\phi_{2\pi}K, K) \to HF^*(K, K)$ is an isomorphism (and in particular hits 1). Next, by the first order piece of the Cardy argument (Proposition @@REF),

$$[\mathcal{CO}_K] \circ [\mathcal{OC}_0] = [\mu^2] \circ [\Delta] : HF^*(\phi_{2\pi}K, K) \to HF^*(K, K) \otimes HF^*(K, K) \to HF^*(K, K). \tag{7.1}$$

But (@@VERIFY) the second map is always an isomorphism, and for the specific model of $[\Delta]$ provided by @@REF, the first map is Serre dual to the multiplication map @@REF:

(Now, unwind the Serre functor, which we can do precisely when there is one critical point, up to shifts, and now count constant triangles). (or rather, just assert that Serre K = K shifted, by the right amount, after which the map is immediately an isomorphism). $\square$

7.2. **Lefschetz-Bott fibrations with one critical point.**

Appendix A. Morita theory for A_∞ localizations

Things we want to say here:

- Lyubashenko show that the localization satisfies a strong universality property with respect to quasi-isomorphisms. But we need a slightly more refined statement, that there is a canonical map in the other direction inducing the quasi-isomorphism.
- define the canonical localization functor on modules and bimodules.
- note that there is a universal morphism from a module/bimodule to the pullback of its localization. This coincides up to equivalence with the universal map on Yoneda modules, diagonal bimodule, graph bimodules, etc.

  We say a module or bimodule is *local* if this morphism is a quasi-isomorphism.
- *locality criterion*: if all quasi-units act by isomorphisms, on both sides, then the bimodule is local.
- The adjoint of localization is fully faithful.

- In particular, Hochschild invariants of local bimodules upstairs coincide with the Hochschild invariants of their localizations downstairs.

A.1. **General nonsense about localization.** There are two types of localization:

- Localization with respect to a class of morphisms $[Z] \subset \mathcal{C}$ (our situation): this produces a category $\mathcal{Q}$ and a functor $L : \mathcal{C} \to \mathcal{Q}$ sending morphisms $[Z]$ to quasi-isomorphisms (L is the (quasi-)universal such).
- Localization with respect to a full subcategory $\mathcal{A}$, often simply called a *quotient*. This produces a new category $\mathcal{Q}$ and a functor $L : \mathcal{C} \to \mathcal{Q}$ in which objects of the subcategory $\mathcal{A}$ are sent to trivial objects (L is again the (quasi-)universal such functor).

The following construction embeds the first setting (ours) into the second: given $[Z]$, let $\mathcal{A} \subset Tw\mathcal{C}$ denote the sub-category consisting of all cones of morphisms in $[Z]$; localizing in the second sense produces a category $\mathcal{Q}'$ with a functor $Tw\mathcal{C} \to \mathcal{Q}'$, and we define $\mathcal{Q}$ to be the image of $\mathcal{C} \hookrightarrow Tw\mathcal{C} \to \mathcal{Q}'$.

In this second setting, there are explicit chain-level formulae for morphisms and compositions in $\mathcal{Q}$ (due to Drinfeld, and in the A_∞ setting, Lyubashenko-Manzyuk); for instance,

$$\hom_{\mathcal{Q}}(K, L) := \mathrm{Cone}(\mathcal{Y}^r_L \otimes_{\mathcal{A}} \mathcal{Y}^l_K \xrightarrow{\mu} \hom_{\mathcal{C}}(K, L)) \tag{A.1}$$

Such formulae are useful for general functorial properties of localization, A_∞ structures, etc., but inconvenient for actually computing morphism spaces, Hochschild invariants, etc. To start, we seek simpler formulae to compute Homs in $\mathcal{F}$ terms of Homs in $\mathcal{O}$ and our geometry.

The first observation, due to [Abouzaid-Seidel], is that, if one is taking the hom of a pair (L_0, L_1) of objects in the correct order ($L_0 > L_1$), then the morphism space $\hom_{\mathcal{O}}(L_0, L_1)$ upstairs computes the right hom downstairs:

Lemma A.1 ("Correct position lemma")**.** *Say $L_0, L_1 \in \mathrm{ob}\ \mathcal{O}^{(I)}$. If $L_0 > L_1$, then*

$$\begin{aligned} &j_I \text{ induces a quasi-isomorphism} \\ &(j_I)_* : CF^*(L_0, L_1) = \hom_{\mathcal{O}_I}(L_0, L_1) \xrightarrow{\sim} \hom_{\mathcal{F}_I}(L_0, L_1). \end{aligned} \tag{A.2}$$

Proof. Quasi-units act on the morphism space upstairs by isomorphisms. . . □

If condition (A.2) is satisfied, we say L_0 is *left local* for L_1 (or L_1 is right local. . .); this lemma gives a criterion. Of course it is not possible to find an object $L_0 \in \mathrm{ob}\ \mathcal{O}$ that is left local for every other object $L_1 \in \mathrm{ob}\ \mathcal{O}$.

Remark A.2. Given an object L_0, one can always find an infinite twisted complex $L_0 \xrightarrow{q} L_0' \xrightarrow{q} L_0'' \xrightarrow{q} \ldots$ (all arrows are quasi-units) that are eventually left local for every object L_1, leading to a direct limit formulations of morphism spaces/products in $\mathcal{F}$. See Abouzaid-Seidel.

We will take a simpler approach, and instead observe the following:

$$\begin{aligned} &\text{If } K_+ \text{ is an object in the sector } \pi/2, \text{ then it is left local} \\ &\text{for every object in the category } \mathcal{O}^0 \text{ (sector 0).} \end{aligned} \tag{A.3}$$

Similarly, K_- is *right local* for every object in $\mathcal{O}^0$. But of course $\mathcal{O}^0$ contains essentially all of the information in $\mathcal{F}_{big}$, so this gives us some handle on most morphisms to or from a specific object.

Insert: -tensor products of bar complexes. -the localization map is always a quasi-isomorphism, so one can compute Hochschild invariants upstairs.

A.2. **The right adjoint of localization.**

Remark A.3. A prototypical example of categorical localization arises in algebraic geometry: If X is a smooth projective variety, $j : U \hookrightarrow X$ is an open sub-variety,

$$Coh(X) \xrightarrow{j^*} Coh(U)$$

presents $Coh(U)$ as the localization of $Coh(X)$ by coherent sheaves $Coh_D(X)$ in X supported along the closed complement $D = X \backslash U$.

The (right) adjoint morphism

$$j_* : Coh(U) \to QCoh(X) = Mod(Coh(X))$$

in general does not preserve coherence, but is a fully faithful embedding, meaning that we can still compute homology/Exts in U after pushing forward to X.

Recall that any functor, $f : \mathcal{C} \to \mathcal{Q}$, localization or not, induces a $\mathcal{Q}-\mathcal{C}$ bimodule Γ_f, the *graph of* f, and a $\mathcal{C}-\mathcal{Q}$ bimodule, the *transposed graph of* f Γ_f^T. Formally, these are defined as one-sided pullbacks of the diagonal bimodule on $\mathcal{Q}$:

$$\Gamma_f := (f, id)^* \mathcal{Q}_\Delta \text{ (so } \Gamma_f(X, Y) := \hom_{\mathcal{Q}}(fX, Y)) \tag{A.4}$$

$$\Gamma_f^T := (id, f)^* \mathcal{Q}_\Delta \text{ (so } \Gamma_f^T(K, L) := \hom_{\mathcal{Q}}(K, fL)), \tag{A.5}$$

which, in turn induce (via convolution/one-sided tensor products), functors between module categories

$$f_* := \cdot \otimes_{\mathcal{C}} \Gamma_f : mod(\mathcal{C}) \to mod(\mathcal{Q}) \tag{A.6}$$

$$f^* := \cdot \otimes_{\mathcal{Q}} \Gamma_f^T : mod(\mathcal{Q}) \to mod(\mathcal{C}) \tag{A.7}$$

and a functor between bimodules categories

$$f_* = (f, f)_* : \Gamma_f^T \otimes_{\mathcal{C}} \otimes \cdot \otimes_{\mathcal{C}} \Gamma_f : \mathcal{C}\text{–mod–}\mathcal{C} \to \mathcal{Q}\text{–mod–}\mathcal{Q} \tag{A.8}$$

$$f^* = (f, f)^* : \Gamma_f \otimes_{\mathcal{Q}} \otimes \cdot \otimes_{\mathcal{Q}} \Gamma_f^T : \mathcal{Q}\text{–mod–}\mathcal{Q} \to \mathcal{C}\text{–mod–}\mathcal{C} \tag{A.9}$$

Remark A.4. The morphism $\Delta_{\mathcal{F}} : \mathcal{F}_{rot} \to \mathcal{Y}^l_{\tilde{K}} \otimes \mathcal{Y}^r_{\tilde{K}}$ could be constructed by applying j_* to the morphism Δ. There is something to check though: that $j_* \mathcal{O}_{rot} \simeq \mathcal{F}_{rot}$ and $j_*(\mathcal{Y}^l_K \otimes \mathcal{Y}^r_{\phi_\pi K}) \simeq \mathcal{Y}^l_{\tilde{K}} \otimes \mathcal{Y}^r_{\tilde{K}}$ for suitable $\tilde{K}$. We basically already checked the first, and omit the second (which is straightforward) here.

Note that there is a simpler definition of pull-back of a module or bimodule, also denoted f^*; this is because these two versions are (quasi-)equivalent via

$$f^* \mathcal{M} \simeq f^*(\mathcal{M} \otimes_{\mathcal{Q}} \mathcal{Q}_\Delta) = \mathcal{M} \otimes_{\mathcal{Q}} \Gamma_f^T.$$

Note that f_* is a *proper functor* (meaning it preserves perfectness), as $f_* \mathcal{Y}^r_K \simeq \mathcal{Y}^r_{fK}$, but f^* may not be.

Lemma A.5 (Technical Localization Lemma)**.** *If f is a localization, then $\Gamma_f \otimes_{\mathcal{C}} \Gamma_f^T \simeq \mathcal{Q}_\Delta$. Namely, the pullback f^* is a fully faithful embedding, with left quasi-inverse given by f_*.*

Proof. Using explicit chain-level models, there is a canonical morphism $\Gamma_f \otimes_{\mathcal{C}} \Gamma_f^T \to \mathcal{Q}_\Delta$. Take the cone, length filter, and apply an explicit contracting homotopy on page 1... □

A.3. **Hochschild invariants of local bimodules.**

Lemma A.6 (Localization Lemma/"Computing upstairs")**.** *If $f : \mathcal{C} \to \mathcal{Q}$ is a localization, and $\mathcal{P}$ is any bimodule over $\mathcal{Q}$, then*

$$f_* : \mathrm{CC}_*(\mathcal{C}, f^*\mathcal{P}) \to \mathrm{CC}_*(\mathcal{Q}, \mathcal{P}) \tag{A.10}$$

is always a quasi-isomorphism.

Proof of Lemma A.6. Let's show that the relevant chain-complexes are quasi-isomorphic via a zig-zag, without worrying about the particular chain map f_*. First, there is a quasi-isomorphism

$$\mathrm{CC}_*(\mathcal{C}, f^*\mathcal{B}) \xleftarrow{\sim} \mathrm{CC}_*(\mathcal{C}, f^*(\mathcal{Q}_\Delta \otimes_{\mathcal{Q}} \mathcal{B} \otimes_{\mathcal{Q}} \mathcal{Q}_\Delta)) = \mathrm{CC}_*(\mathcal{C}, \Gamma_f^T \otimes_{\mathcal{Q}} \mathcal{B} \otimes_{\mathcal{Q}} \Gamma_f) \tag{A.11}$$

coming from an underlying quasi-isomorphism of bimodules. Next, rewrite the latter complex as a two-sided tensor product

$$\mathrm{CC}_*(\mathcal{C}, \Gamma_f^T \otimes_{\mathcal{Q}} \mathcal{B} \otimes_{\mathcal{Q}} \Gamma_f) = (\Gamma_f \otimes_{\mathcal{C}} \Gamma_j^T) \otimes_{\mathcal{Q}\text{–}\mathcal{Q}} \mathcal{B}. \tag{A.12}$$

Lemma A.5 gives a quasi-isomorphism

$$(\Gamma_f \otimes_{\mathcal{C}} \Gamma_j^T) \otimes_{\mathcal{Q}\text{–}\mathcal{Q}} \mathcal{B} \xrightarrow{\sim} \mathcal{Q}_\Delta \otimes_{\mathcal{Q}\text{–}\mathcal{Q}} \mathcal{B} \tag{A.13}$$

But the right complex is quasi-isomorphic to $\mathrm{CC}_*(\mathcal{Q}, \mathcal{B})$ via explicit maps (for instance, thinking of it as the the Hochschild complex of $\mathcal{Q}_\Delta \otimes_{\mathcal{Q}} \mathcal{B}$). □

Remark A.7 (General nonsense about locality). If $f : \mathcal{C} \to \mathcal{Q}$ is a localization, and $\mathcal{B}$ is any module/bimodule over $\mathcal{C}$, we say that $\mathcal{B}$ is **local** if $\mathcal{B} \simeq j^*\tilde{B}$, for some $\mathcal{Q}$-bimodule $\tilde{B}$. It is a general fact that there is always a morphism $\mathcal{B} \to j^*j_*\mathcal{B}$, which need not be a quasi-isomorphism, but is the universal map to j^* of a bimodule. In particular, if $\mathcal{B} \simeq j^*\tilde{\mathcal{B}}$ for some bimodule $\tilde{\mathcal{B}}$, then $\tilde{\mathcal{B}} \simeq j_*\mathcal{B}$. So we call $j_*\mathcal{B}$ the **localization** of $\mathcal{B}$.

There is a straightforward criterion for when a module/bimodule $\mathcal{B}$ is *local*, meaning that it is quasi-isomorphic to $j^*j_*\mathcal{B}$: if multiplication by quasi-units act (on the left and/or right) by quasi-isomorphisms.

Appendix B. C^0 estimates

B.1. **Maximum principle for admissible Lagrangians.** Let Σ be the complement of finitely many points on the boundary of a compact Riemann surface, and assume that the boundary components are labelled by admissible Lagrangians in E. Fix (positive or negative) strip-like ends at all marked points. Choose maps

$$H\colon\ \Sigma \to \mathcal{H} \tag{B.1}$$

$$J\colon\ \Sigma \to \mathcal{J} \tag{B.2}$$

such that (H, J) are constant on the ends. In particular, we can associate to each end ϵ a space of Hamiltonian chords $\mathcal{X}_\epsilon$ (i.e. flow lines $[0,1] \to E$ of X_H) between the Lagrangians labelling its boundary. We assume:

(B.3) $\mathcal{X}_\epsilon$ consists of finitely many elements all of which are non-degenerate and disjoint from the boundary.

We the consider the space of finite energy maps

$$u\colon\ \Sigma \to E \tag{B.4}$$

which solve the differential equation

$$(du - \alpha \otimes X_H)^{0,1} = 0 \tag{B.5}$$

with α a 1-form on Σ. Letting $\lambda\colon \Sigma \to \mathbb{R}$ be the function which records the vertical slope of H, we say that the above equation is *vertically monotone* if

$$d(\lambda \cdot \alpha) \geq 0 \tag{B.6}$$

as a 2-form on Σ (positivity is measured with respect to the standard orientation induced by the complex structure).

Lemma B.1. *Every solution to Equation* (B.5) *which intersects the boundary is contained in it.*

Proof. We apply the maximum principle for ρ. At the interior, this is standard. At the boundary, this is the maximum principle with Neumann boundary conditions. □

Let $b\colon \Sigma \to \mathbb{R}$ be the function which records to horizontal slope of H. We assume that the pseudo-holomorphic equation is *horizontally monotone* in the sense that

$$d(b \cdot \alpha) \geq 0 \tag{B.7}$$

Let $K \subset \mathbb{C}$ be a compact subset such that (i) all chords corresponding to the ends of Σ project to E, (ii) π is holomorphic outside of K, and (iii) for each $z \in \Sigma$, $H_z = \lambda\rho + b_z r^2$ outside the inverse image of K. Applying the maximum principle for Neumann boundary conditions to the function $\pi \circ r$, we find that

Lemma B.2. *All solutions to Equation* (B.5) *have image contained in* $\pi^{-1}K$.

eed to find
mething that
orks both for
agonal and
r products

B.2. **C^0-estimates for correspondences.** Let E^{2k} be a product of an even number of copies of E, equipped with the product symplectic form which, on the i^{th} factor, is $(-1)^i\omega$. This is a manifold with corners, with top dimensional strata given by $E^{i-1} \times \partial E \times E^{2k-i}$. Let $\rho_{E^{2k}}$ denote the sum of the pullbacks of the functions ρ on each factor.

Definition B.3. *An* admissible correspondence *is a Lagrangian* $\Gamma \subset E^{2k}$ *if* $d\rho_{E^{2k}}|\Gamma$ *does not vanish in a neighbourhood of the boundary, and* $d^c\rho_{E^{2k}}|\Gamma$ *identically vanishes.*

We denote by $\mathcal{H}_0(E^{2k})$ the space of functions on E^{2k} with compact support in the interior of E^{2k}, and by $\mathcal{J}(E^{2k})$ the space of tame almost complex structures which, outside a compact set, agree with a product of almost complex structures $J_i \in \mathcal{J}$.

Let Σ be the complement of finitely many points in a Reimann surface with boundary, and assume that the boundary components are labelled by admissible $\Gamma_i \subset E^{2k}$ which do not intersect near the boundary. Fix (positive or negative) strip-like ends at all interior and boundary marked points. Choose maps

$$H\colon \Sigma \to \mathcal{H}_0(E^{2k}) \tag{B.8}$$

$$J\colon \Sigma \to \mathcal{J}(E^{2k} \tag{B.9}$$

such that (H, J) are constant on the ends. Let

$$u\colon \Sigma \to E \tag{B.10}$$

be a finite energy map solving the differential equation

$$(du - \alpha \otimes X_H)^{0,1} = 0 \tag{B.11}$$

with α a 1-form on Σ.

Lemma B.4. *Every solution to Equation* (B.11) *which intersects the boundary is contained in it.*

Proof. Let σ denote a stratum of ∂E^{2k}, and let ρ_σ denote the sum of the functions ρ_i associated to each top-dimensional stratum adjacent to σ. By assumption, each correspondence satisfies Neumann boundary conditions with respect to ρ_σ in a neighbourhood of σ. □

Lemma B.5. *If the distance between all pairs* $(\pi\Gamma_i, \pi\Gamma_j)$ *is bounded above outside a compact set, any finite energy solution to Equation* (B.11) *is contained in a compact subset of* E^{2k}.

Proof. We have assumed that each Lagrangians $\pi\Gamma_i$ is geometrically bounded, and the assumption that the distance is bounded implies that their union is geometrically bounded (outside a compact set). Since the projection from $E^{2k} \to \mathbb{C}^{2k}$ is holomorphic outside a proper subset, the result follows from the monotonicity Lemma [???]. □

Combining these two results, we have

Proposition B.6. *All bounded energy solutions to Equation* (B.11) *are contained in a compact subset of the interior of* E^{2k}. □

B.3. **Gauge transformations and doubling.** Consider $H \in \mathcal{H}_{\lambda,b}$, and write ϕ_t for its Hamiltonian flow. Let $J_{s,t}\mathcal{J}$ be a family of almost complex structures parametrised by the strip $(s,t) \in \mathbb{R} \times [0,1]$. This determines the holomorphic curve equation

$$J_{s,t}\partial_s u = \partial_t u - X_H \tag{B.12}$$

on the space of maps $u \to E$.

Define $\tilde{J}_{s,t} = (\phi_t^* J_{s,t}, -\phi_{-t}^* J_{s,1-t}))$ for $(s,t) \in \mathbb{R} \times [0,1/2]$. Since the Hamiltonian flow of H preserves holomorphicity of π outside a compact set, we have $\tilde{J}_{s,t} \in \mathcal{J}(E^2)$. Given a solution to Equation (B.12), the map

$$\phi^* u \colon \mathbb{R} \times [0,1/2] \to E^2 \tag{B.13}$$

$$(s,t) \mapsto (\phi_t(u(s,t)), \phi_{-t}(u(s,1-t))) \tag{B.14}$$

is holomorphic with respect to $\tilde{J}_{s,t}$. Let $\Delta_\phi \subset E^2$ denote the graph of ϕ^{-1}, i.e. point $(x, \phi^{-1}x)$.

Double check all indices are right

Lemma B.7. *The map* $u \to \phi^* u$ *gives a bijective correspondence between solutions to Equation* (B.12) *with boundary conditions* L *at* $t=0$ *and* K *at* $t=1$ *and solutions to the holomorphic curve equation with boundary* $L \times K$ *at* $t=0$ *and* Δ_ϕ *at* $t=1/2$. □

There is a straightforward generalisation of the above result for multiple Lagrangians: If $\{K_i\}_{i=1}^r$ and $\{L_j\}_{j=1}^s$ are Lagrangians, and $\{I_i\}_{i=1}^r$ and $\{J_j\}_{j=1}^s$ are subdivision of the real line into intervals, then the above construction yields a bijective correspondence between (i) solutions to Equation (B.12) with boundary condition L_i on $I_i \times \{0\}$ and K_j on $J_j \times \{1\}$ and (ii) $\tilde{J}_{s,t}$-holomorphic maps from $\mathbb{R} \times [0,1/2]$ to E^2 with boundary conditions $L_i \times K_j$ on $(I_i \cap J_j) \times \{0\}$, and Δ_ϕ on $\mathbb{R} \times \{1/2\}$.

B.4. **Plurisubharmonic functions on $\mathbb{C}^2$. TODO 9/2015: Revise the below information to keep the maximum principle and remove the Floer theory, and maybe need to use the fact that the maximum of two plurisubharmonic functions is plurisubharmonic in proof.**

Remark: Cieliebak-Eliashberg Chapter 3 reviews the fact that the max of two plurisubharmonic functions, smoothed in a certain way, can be made plurisubharmonic.

$$CF^*(\Delta, \Delta_\sharp). \tag{B.15}$$

Note that this is easy to set up because $\Delta_\sharp$ is radially invariant, so the maximum principle for $|z_1|^2 + |z_2|^2$ (as a function on $\mathbb{C}^2$) can be used.

If L and K are admissible, it is easy to define $HF^*(K, L)$ by standard methods (compactly supported Hamiltonian perturbations, and maximum principle for holomorphic curves) if the directions at infinity are disjoint. It is also easy to define this Floer group if $K = L$ (use a proper Morse function). This means that we have Floer cohomology groups $HF^*(\phi_\sharp K, L)$ whenever the angular directions satisfy some inequalities.

Lemma B.8. *Assume that $A_K \subset [-\pi + 4\epsilon, -2\epsilon]$ and $A_L \subset [-\epsilon, \epsilon]$. We have natural isomorphisms:*

$$HF^*(\phi_\sharp K, L) = HF^*(\phi_\pi K, L) = HF^*(K \times L, \Delta_\pi) \cong HF^*(K \times L, \Delta_\sharp) \tag{B.16}$$

If $A_{L'} \subset [-\epsilon, \epsilon]$, we have natural isomorphisms:

$$HF^*(\phi_\sharp L', L) = HF^*(\phi_{2\pi+2\epsilon} L', L) = HF^*(L' \times L, \Delta_{2\pi+2\epsilon}) \cong HF^*(L' \times L, \Delta_\sharp). \tag{B.17}$$

Proof. We need to make sure that the second two groups are well defined. The problem is that $K \times L$, is not radially invariant. However, we can construct a (weakly) psh function ψ, which vanishes in a compact set and agrees with $|z|^2$ outside a compact set. We can then apply the maximum principle for $\psi(z_1) + \psi(z_2)$ when computing the Floer groups of these Lagrangians with $\Delta_\sharp$.

The first equality follows from $\phi_\sharp K = \phi_\pi K$. The second from “doubling,” and the third from either the “integrated maximum principle” or from the existence of a Hamiltonian isotopy between Δ_π and $\Delta_\sharp$), among radially invariant Lagrangians, so that no intersection points with $K \times L$ are introduced. □

Remark B.9. It is delicate to directly use the doubling idea for $\phi_\sharp$ because it is not holomorphic outside a compact set.

References

[AA] Mohammed Abouzaid and Denus Auroux. In preparation.

[A] Mohammed Abouzaid, *A geometric criterion for generating the Fukaya category*, Publ. Math. Inst. Hautes Études Sci. **112** (2010), 191–240. MR2737980 (2012h:53192)

[AAK] Mohammed Abouzaid, Denis Auroux, and Ludmil Katzarkov, *Lagrangian fibrations on blowups of toric varieties and mirror symmetry for hypersurfaces* (2012).

[A] D. Auroux, *Mirror symmetry and T-duality in the complement of an anticanonical divisor*, J. Gökova Geom. Topol. **1** (2007), 51–91.

[AFO$^+$] M. Abouzaid, K. Fukaya, Y.-G. Oh, H. Ohta, and K. Ono, *Quantum cohomology and split generation in Lagrangian Floer theory.* in preparation.

[AS] Mohammed Abouzaid and Paul Seidel, *Lefshetz fibration techniques in wrapped Floer cohomology.* In preparation.

[BC1] Paul Biran and Octav Cornea, *Lagrangian cobordism. I*, J. Amer. Math. Soc. **26** (2013), no. 2, 295–340.

[BC2] ———, *Lagrangian cobordism and Fukaya categories*, Geom. Funct. Anal. **24** (2014), no. 6, 1731–1830.

[BC3] ———, *Lagrangian cobordism in lefschetz fibrations* (2015). available at `http://arxiv.org/abs/1504.00922`.

[HV] Kentaro Hori and Cumrun Vafa, *Mirror symmetry.* available at `http://arxiv.org/abs/hep-th/0002222`.

[K1] Maxim Kontsevich, *Homological algebra of mirror symmetry*, Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Zürich, 1994), 1995, pp. 120–139. MR1403918 (97f:32040)

[K2] ———, *Lectures at ENS Paris, Spring 1998*, 1998. set of notes taken by J. Bellaiche, J.-F. Dat, I. Marin, G. Racinet and H. Randriambololona.

[K3] ———, *Symplectic geometry of homological algebra*, 2013. available at `http://www.ihes.fr/~maxim/TEXTS/Symplectic_AT2009.pdf`.

[S1] P. Seidel, *Vanishing cycles and mutation*, Proceedings of the 3rd european congress of mathematics (barcelona, 2000), 2001, pp. 65–85.

[S2] Paul Seidel, *Fukaya categories and Picard-Lefschetz theory*, Zurich Lectures in Advanced Mathematics, European Mathematical Society (EMS), Zürich, 2008. MR2441780 (2009f:53143)

[S3] ———, *Homological mirror symmetry for the genus two curve*, J. Algebraic Geom. **20** (2011), no. 4, 727–769.

[S4] ———, *Fukaya A_∞-structures associated to Lefschetz fibrations. I*, J. Symplectic Geom. **10** (2012), no. 3, 325–388.

[S5] ———, *Fukaya A_∞ structures associated to Lefschetz fibrations. II* (2014). available at `http://arxiv.org/abs/1404.1352`.

[S6] ———, *Fukaya A_∞-structures associated to Lefschetz fibrations. II 1/2* (2015). available at `http://arxiv.org/abs/1504.06317`.