X a space, $p,q \in X$.

- Recall a **path** γ from p to q is a continuous map $\gamma : \overset{I}{\overset{\shortparallel}{[0,1]}} \longrightarrow X$ with $\gamma(0) = p$, $\gamma(1) = q$.

Def: Two paths γ_0, γ_1 from p to q are <u>homotopic</u> (<u>rel. endpoints</u>) if there is a continuous map

$$(*) \quad H : \underbrace{[0,1]}_{s \text{ (path param.)}} \times \underbrace{[0,1]}_{t \text{ (time param.)}} \longrightarrow X$$

[Note: H is called a <u>homotopy</u> from γ_0 to γ_1.]

satisfying: $H(0,t) = p$, $H(1,t) = q$, $H(s,0) = \gamma_0(s)$, $H(s,1) = \gamma_1(s)$.

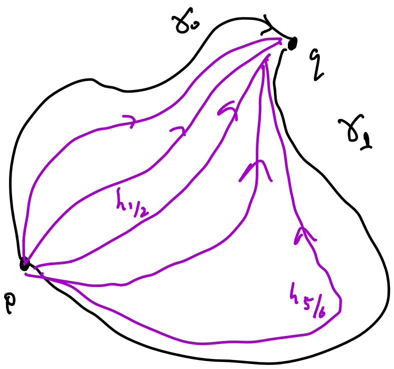
Put another way if there is a continuous H as in $(*)$, such that, if we call

$$h_t = H(-,t) : I \longrightarrow X,$$

then each h_t is a path in X from p to q and $h_0 = \gamma_0$, $h_1 = \gamma_1$.

Sometimes we use $\{h_t\}_{t \in [0,1]}$ to refer to the homotopy; but it's very important H be continuous!

Pictorially:



Write $\gamma_0 \simeq \gamma_1$ if γ_0, γ_1 are homotopic paths (rel. endpoints).

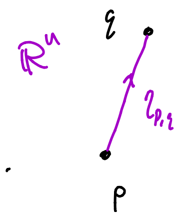
<u>Lem</u> (w/o proof): For any p,q,

(1) $\simeq$ is an equivalence relation on the set of paths from p to q.

(e.g., if $\gamma_0 \simeq \gamma_1$ & H is a homotopy from γ_0 to γ_1, w/ $H(-,t) = h_t$, then $(s,t) \mapsto H(s, 1-t)$ is (e.g., h_{1-t}) a homotopy γ_1 to γ_0, so $\gamma_1 \simeq \gamma_0$ too).

<u>Example</u>: (1) $X = \mathbb{R}^n$, $\vec{p}, \vec{q} \in X$ any two points.

Then, all paths $\vec{\gamma}_0, \vec{\gamma}_1$ from $\vec{p}$ to $\vec{q}$ are homotopic (rel endpoints).



<u>Pf</u>: We'll just show any path $\vec{\gamma}$ is htopic to the <u>straight-line path</u> $\vec{\eta}_{p,q}(s) = s\vec{q} + (1-s)\vec{p}$ (↑ scalar-vec. mult.)

note that $H(s,t) = t\vec{\gamma}(s) + (1-t)\eta_{p,q}$

is (a) continuous (b) $H(0,t) = t\vec{\gamma}(0) + (1-t)\vec{\eta}_{p,q}(0) = t\vec{p} + (1-t)\vec{p} = \vec{p}$

$H(1,t) = \vec{q}$

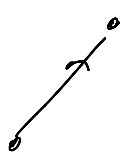
(c) $\underset{h_0}{\underset{\shortparallel}{H(\cdot,0)}} = \vec{\eta}_{p,q}$ (d) $H(\cdot,1) = \vec{\gamma}$.

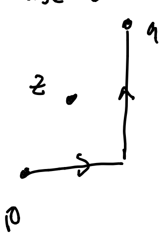
(2) $X = \mathbb{R}^n$, z any point, p, q any other two points not equal to z, $\boxed{n>1}$

Then, any path γ from $p \to q$ is homotopic (rel. endpts.) to a path in $\mathbb{R}^n$ which avoids z (doesn't have z in image).

[False $n=1$. Ex $z = 0 \in \mathbb{R}$. any path $-1 \to 1$ has to pass through 0].

Sketch: (a) $\exists$ a path τ from p to q avoiding any point z. (we showed this earlier in class: if straight line path avoids z, done, otherwise use a "step" path.)





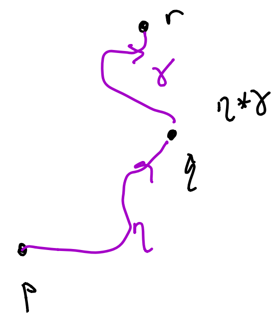
(b) τ is homotopic to γ by (1).

Concatenation:

If $\eta : [0,1] \to X$ is a path from p to q and $\gamma : [0,1] \to X$ is a path from q to r, then can define a new path from p to r, the concatenation

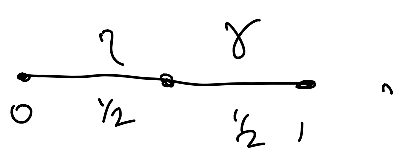
$$\eta * \gamma : [0,1] \longrightarrow X$$

$$\eta * \gamma(t) = \begin{cases} \eta(2t) & 0 \le t \le 1/2 \\ \gamma(2t-1) & \frac{1}{2} \le t \le 1. \end{cases}$$



(why continuous? pasting lemma)

"follow η, then γ, each at double speed"



Constant paths: There is a canonical path from p to p, called the constant path at p.

$$c_p : [0,1] \longrightarrow X$$
$$t \longmapsto p.$$

Reversing paths: Given a path $\gamma : [0,1] \longrightarrow X$ from p to q, the reverse of γ

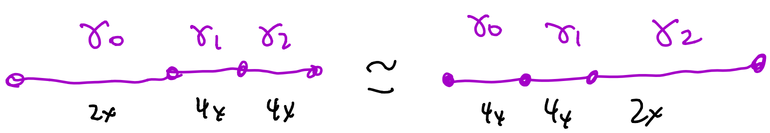
$$\bar{\gamma} : [0,1] \longrightarrow X$$
$$t \longmapsto \gamma(1-t)$$

, is a path from q to p. (why is it continuous if γ is?)

Key Properties of these operations (w/o proof): Recall $\simeq$ means "are homotopic rel. endpoints".

(2) • For any triple $\gamma_0, \gamma_1, \gamma_2$ of a path from $p \to q$, from $q \to r$, and from $r \to s$ respectively,

$$\gamma_0 * (\gamma_1 * \gamma_2) \simeq (\gamma_0 * \gamma_1) * \gamma_2 .$$

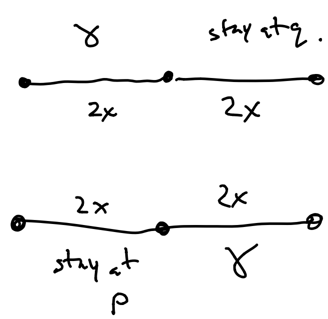


(3) • If γ is a path from p to q, then

$$\gamma * c_q \simeq \gamma$$

const. path at q

$$\& \quad c_p * \gamma \simeq \gamma$$

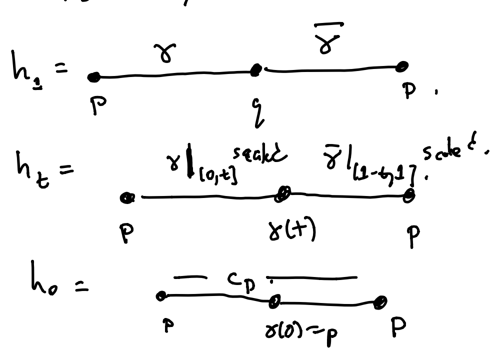


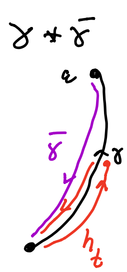
(4) • If γ any path from p to q, & $\bar{\gamma}$ its reverse, then

$$\gamma * \bar{\gamma} \simeq c_p .$$

a path p to p

sketch:





and $\bar{\gamma} * \gamma \simeq c_q$

a path q to q.

(5) If $\gamma_0 \simeq \gamma_1$, then $\gamma_0 * \eta \simeq \gamma_1 * \eta$, and $\tau * \gamma_0 \simeq \tau * \gamma_1$.

(6) $f: X \longrightarrow Y$ any continuous map, and γ, η paths from p to q & q to r, then $f\circ\gamma$, $f\circ\eta$ are paths from $f(p)$ to $f(q)$ & $f(q)$ to $f(r)$.

& $f\circ(\gamma * \eta) = (f\circ\gamma) * (f\circ\eta)$.

(7) If $\gamma_0 \simeq \gamma_1$, then $f\circ\gamma_0 \simeq f\circ\gamma_1$.

Def: X space, $p \in X$ point.

- $\pi_1(X,p) := \{\text{paths } \gamma \text{ from } p \text{ to } p\} / \simeq$ ← homotopy equiv. rel. endpoints.

- $\times : \pi_1(X,p) \times \pi_1(X,p) \longrightarrow \pi_1(X,p)$

  $[\gamma] \times [\eta] := [\gamma * \eta]$. Check: • well-defined (8)

Thm: • $\pi_1(X,p)$ is a group; meaning

- $\times$ is associative (from (2))
- there is an identity element e (so $e \times g = g \times e = g \ \forall g$):

  $e = [c_p]$ works by (3).

- there are inverses: for each $\alpha \in \pi_1(X,p)$, $\exists$ "α^{-1}" w/ $\alpha \times \alpha^{-1} = \alpha^{-1} \times \alpha = e$.

  If $\alpha = [\gamma]$, then $\alpha^{-1} = [\bar{\gamma}]$ works.

Lecture end

Thm: $f: X \xrightarrow{\text{cont.}} Y$ induces a group homomorphism

$$f_*: \pi_1(X,p) \longrightarrow \pi_1(Y, f(p))$$
$$[\gamma] \longmapsto [f\circ\gamma]$$

(well-defined?) (sends e to e?)

($f_*(g \times h) = f_*(g) \times f_*(h)$)?

Use (6) + (7).

& $f_* \circ g_* = (f\circ g)_*$ (9)

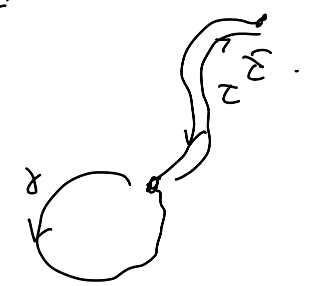
Cor: If $f: X \underset{\text{homeo}}{\overset{\cong}{\to}} Y$, then $f_*: \pi_1(X,p) \underset{\text{group iso.}}{\overset{\cong}{\longrightarrow}} \pi_1(Y, f(p))$. [Invariance].

Thm: If p, q in the same path component then $\pi_1(X,p) \xrightarrow[\text{group iso.}]{\cong} \pi_1(X,q)$.

Pf: Let τ be any path p to q.

Define $h([\gamma])$ by $[\bar{\tau} * \gamma * \tau]$.

$q \to p \to p \to q$.



check h is a group isomorphism...

In $\mathbb{R}^n$, any path p to p is homotopic to the constant path c_p,

so $\pi_1(\mathbb{R}^n, p) = \{e\}$.
any p

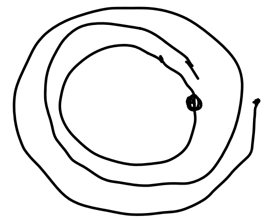
Recall $S^n = \{x_1^2 + \cdots + x_{n+1}^2 = 1\} \subset \mathbb{R}^{n+1}$.

Claims: (1) $\pi_1(S^1) \cong (\mathbb{Z}, +)$. we'll drop basepoint from notation, but use basepoint $(1,0)$.

(2) $\pi_1(S^n) = \{e\}$, $n > 1$. (on HW).

In case (1), for each $n \in \mathbb{Z}$, here is a representative of a path from $(1,0)$ to $(1,0)$:
"wind n-times counterclockwise"

$$\gamma_n : [0,1] \longrightarrow S^1$$
$$t \longmapsto (\cos 2\pi n t, \sin 2\pi n t)$$



Note: $\gamma_0 = \text{Const}_{(1,0)}$.

Have to check: (a) each path from $(1,0)$ to $(1,0)$ is homotopic to a unique γ_n.

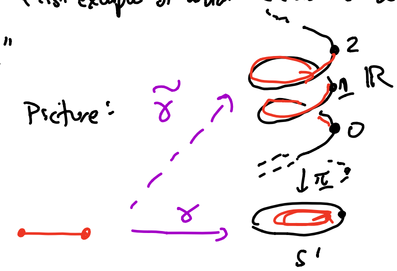
(b) $\gamma_m * \gamma_n \simeq \gamma_{m+n}$. (exercise: note $\gamma_m * \gamma_n$ full circle, $\frac{1}{2}(\frac{1}{n})$, $\frac{1}{2}(\frac{1}{m})$, $1/2$, full circle, and $\gamma_{m+n} = \frac{1}{m+n}$, full circle)

Consider $\mathbb{R} \xrightarrow[(\cos 2\pi\theta, \sin 2\pi\theta)]{\pi} S^1$. First example of what we call a "covering space."

$0 \longmapsto (1,0) \leftrightarrow$ "1"

$n \in \mathbb{Z} \longmapsto$

Picture:



Fact: (exercise):

(1) Given a path $\gamma : [0,1] \longrightarrow S^1$ with both endpoints at "1", there is a unique lift of γ, $\tilde{\gamma} : [0,1] \longrightarrow \mathbb{R}$ (meaning $\pi \circ \tilde{\gamma} = \gamma$), with $\tilde{\gamma}(0) = 0$.

(Know $\tilde{\gamma}(0)$ projects to 0, so $\tilde{\gamma}(0)$, $\tilde{\gamma}(1)$)

(2) γ_0 and γ_1 are homotopic rel. endpoints iff their lifts $\tilde{\gamma}_0$ and $\tilde{\gamma}_1$ are.

Proof of (1): Note that given any $\gamma : [0,1] \longrightarrow S^1$ path from "1" to "1";

$\exists$ a unique lift $\tilde{\gamma}$ path from 0 to some other preimage $n \in \mathbb{Z}$ of "1".

Up to homotopy, $\exists !$ path from 0 to n in $\mathbb{R}$, $\tilde{\gamma}_n$; so $\tilde{\gamma}$ is homotopic to $\tilde{\gamma}_n$;

so by (2), γ is homotopic to γ_n.

Next, note that γ_n are all nonhomotopic (rel. endpts.) b/c $\tilde{\gamma}_n$ are.

So, as sets, $\pi_1(S^1) \cong \mathbb{Z}$.

$$[\gamma_n] \longleftarrow n$$

Exercise: prove this as groups.