Math 440 Homework 1

Due Friday, Sept. 1, 2017 by 4 pm

Please remember to write down your name on your assignment.

Please submit your homework to our TA Viktor Kleen, either in his mailbox (in KAP 405) or under the door of his office (KAP 413). You may also e-mail your solutions to Viktor provided:

- you have typed your homework solutions; or
- you are able to produce a very high quality scanned PDF (no photos please!),

Note: Problems 4 and 5 below will require you to read Munkres §1.3, the subsection titled **Equivalence relations.**

1. Prove or disprove each of the following equalities for sets A, B, C, (and possible D). Namely, either prove that the equality always holds, or if it doesn't always hold, clearly indicate with a counterexample which inclusion fails (and then either prove the other inclusion holds, or give a counterexample to it holding too).
   a) $A \cap (B - C) = (A \cap B) - (A \cap C)$.

   b) $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$.

   c) $A - (B - C) = (A - B) \cup (A \cap C)$.

2. Prove or disprove each of the following statements. Namely, either prove that the property always holds, or provide an explicit example where it is false.
   a) If $f : X \to Y$ and $A, A' \subset X$, then $f(A \cap A') = f(A) \cap f(A')$.

   b) If $f : X \to Y$ and $A, A' \subset X$, then $f(A \cup A') = f(A) \cup f(A')$.

   c) If $f : X \to Y$ and $B, B' \subset Y$, then $f^{-1}(B \cap B') = f^{-1}(B) \cap f^{-1}(B')$.

3. For each of the following statements of the form "If P then Q", write its inverse[1], coverse, and contrapositive statements. State which versions of the statement are true, and conclude whether P and Q are equivalent or not.

   a) If a function $f : \mathbb{R} \to \mathbb{R}$ is continuous at a, then it is differentiable at a.

   b) If $x < 0$, then $x^2 - x > 0$.

[1] In class, we referred to the statement *if not*(P) *then not*(Q) the "negation" of *if* P *then* Q, but this was misleading terminology. Strictly speaking, it is *not* logically the negation of *if* P *then* Q; the logical negation is "P *and not*(Q)". Henceforth, we will refer to *"if not*(P) *then not*(Q)*"* as the *inverse* of "*if* P *then* Q".

4. Write the negation of the following sentence (note: be careful when negating statements involving there exists and for all!). State whether either statement and its negation are true.
   a) For every number x, there is at least one number y such that $e^y = x$. (or more formally, "for all $x \in \mathbb{R}$, there exists $y \in \mathbb{R}$ such that $e^y = x$.")

   b) There is at least one number a in $[0, 1]$ such that the derivative of x^3 at $x = a$ is 0. (more formally, "There exists $a \in [0, 1]$ such that the derivative of x^3 at $x = a$ is 0").

   c) For every function $f : [0, 1] \to [0, 1]$ which is continuous and strictly increasing on $[0, 1]$, there is a unique function $g : [0, 1] \to [0, 1]$ such that $g(f(x)) = x$ for every $x \in [0, 1]$.

5. **Equivalence relations I**: Solve Munkres §1.3, Exercise 1. **Note**: as mentioned in the beginning, you'll need to read Equivalence relations before you attempt this problem or the next problem.

6. **Equivalence relations II**: For each of the following relations defined on $\mathbb{R}$, the set of all real numbers, check whether the relation is reflexive, symmetric, and transitive (prove the property is true, or find a counterexample). Conclude whether each of the relations is an equivalence relation.
   a) $a \sim b$ means $ab \neq 0$.

   b) $a \sim b$ means $ab \geq 0$.

   c) $a \sim b$ means $\lfloor a \rfloor = \lfloor b \rfloor$ (where $\lfloor x \rfloor$ denotes the largest integer $\leq x$, the "floor" of x).

7. a) Let A, B be sets. Show that there is a bijection between $A \times B$ and $B \times A$.

   b) Let A and B be sets. In class, we defined the *mapping set*

   $$Maps(A, B) = \{f : A \to B\}$$

   to be the set whose elements are distinct functions $f : A \to B$ (two functions f_1 and f_2 are equal if $f_1(a) = f_2(a)$ for every $a \in A$).

   Let X be a set. In class, we defined, for a natural number n, the product X^n to be the set of ordered n-tuples of elements of X.

   $$X^n = \{(x_1, \ldots, x_n) | x_i \in X\}.$$

   Prove that there is a natural bijection

   $$X^n \cong Maps(\{1, \ldots, n\}, X).$$

   (In fact, Munkres §1.5 simply defines X^n to be $Maps(\{1, \ldots, n\}, X)$!).

   c) In light of the previous part, let's define

   $$\text{“}X^\infty\text{”} = Maps(\mathbb{N}, X)$$

Equivalently (as in part (b)), "X^∞" can be described as the set of infinite-length tuples $(x_1, x_2, \ldots,)$. In fact, the notation "X^∞" is a little imprecise—as there are many non-equivalent infinite-sized sets (for instance, $\mathbb{R}$ and $\mathbb{N}$). In contrast, all sets with n elements are equivalent[2]. So, instead, we will use the notation

$$X^\omega = \text{“}X^\infty\text{”} = Maps(\mathbb{N}, X)$$

(where ω, the Greek letter "omega", is typically used to indicate the "size", or *cardinality*, of $\mathbb{N}$).

If X is a non-empty set, and n any natural number, find a bijective map $f : X^n \times X^\omega \to X^\omega$.

[2]Two sets A and B are *equivalent* if one can find a bijection $A \xrightarrow{\cong} B$.