Homework 1

Exercise 1.1. Prove or disprove each of the following equalities for sets A, B, C, (and possibly D). Namely, either prove that the equality always holds, or if it doesn't always hold, clearly indicate with a counterexample which inclusion fails (and then either prove the other inclusion holds, or give a counterexample to it holding too).

(i) $A \cap (B \setminus C) = (A \cap B) \setminus (A \cap C)$
(ii) $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$
(iii) $A \setminus (B \setminus C) = (A \setminus B) \cup (A \cap C)$

Solution.

(i) Take any $x \in A \cap (B \setminus C)$, that is, $x \in A$ and $x \in B$ but $x \notin C$. Then we have $x \in A \cap B$ and $x \notin A \cap C$ and therefore $x \in (A \cap B) \setminus (A \cap C)$.
Conversely, take any $x \in (A \cap B) \setminus (A \cap C)$, that is, $x \in A$ and $x \in B$ but $x \notin A \cap C$. Because $x \in A$, the latter implies that $x \notin C$ and therefore $x \in A \cap (B \setminus C)$. Therefore the equality holds.

(ii) Suppose $(x, y) \in (A \times B) \cap (C \times D)$. Then $x \in A$, $x \in C$, $y \in B$ and $y \in D$. Consequently, $x \in A \cap C$ and $y \in B \cap D$ which means $(x, y) \in (A \cap C) \times (B \cap D)$.
Conversely, if $(x, y) \in (A \cap C) \times (B \cap D)$, then $x \in A \cap C$ and $y \in B \cap D$. This means $x \in A$ and $x \in C$ while $y \in B$ and $y \in D$, i. e. $(x, y) \in A \times B$ and $(x, y) \in C \times D$. But then $(x, y) \in (A \times B) \cap (C \times D)$, so the equality holds.

(iii) Suppose $x \in A \setminus (B \setminus C)$. Then $x \in A$ but $x \notin B \setminus C$, which means $x \notin B$ or $x \in C$. So we have $x \in A$ and $x \notin B$ or $x \in A$ and $x \in C$. Written differently, this means $x \in (A \setminus B) \cup (A \cap C)$.
Conversely, let $x \in (A \setminus B) \cup (A \cap C)$, that is, $x \in A$ but $x \notin B$ or $x \in A$ and $x \in C$. In other words, $x \in A$ and either $x \notin B$ or $x \in C$. So, $x \in A$ and $x \notin B \setminus C$ as before. We conclude $x \in A \setminus (B \setminus C)$ and therefore the equality holds.

Exercise 1.2. Prove or disprove each of the following statements. Namely, either prove that the property always holds, or provide an explicit example where it is false.

(i) If $f : X \longrightarrow Y$ and $A, A' \subset X$, then $f(A \cap A') = f(A) \cap f(A')$.
(ii) If $f : X \longrightarrow Y$ and $A, A' \subset X$, then $f(A \cup A') = f(A) \cup f(A')$.
(iii) If $f : X \longrightarrow Y$ and $B, B' \subset Y$, then $f^{-1}(B \cap B') = f^{-1}(B) \cap f^{-1}(B')$.

Solution.

(i) Consider the function $f : \{1, 2\} \longrightarrow \{3\}$ with $f(1) = f(2) = 3$. Then $f(\{1\} \cap \{2\}) = f(\emptyset) = \emptyset$ while $f(\{1\}) \cap f(\{2\}) = \{3\} \neq \emptyset$. So this property does not hold in general.

(ii) Suppose $x \in f(A \cup A')$. This means that x is the image of some $y \in A \cup A'$; that is $f(y) = x$ for some $y \in A \cup A'$. Now, if $y \in A$, then $x = f(y) \in f(A)$, and if $y \in A'$, then $x = f(y) \in f(A')$. In either case $x \in f(A) \cup f(A')$. Hence, $f(A \cup A') \subset f(A) \cup f(A')$.
Conversely, suppose $x \in f(A) \cup f(A')$. If $x \in f(A)$, then $x = f(y)$ for some $y \in A$. If on the other hand $x \in f(A')$, then $x = f(y)$ for some $y \in A$. In either case, there is some $y \in A \cup A'$ with $x = f(y)$. But this just means that $x \in f(A \cup A')$. Hence, $f(A) \cup f(A') \subset f(A \cup A')$ and we conclude that in fact $f(A \cup A') = f(A) \cup f(A')$.

(iii) Suppose that $x \in f^{-1}(B \cap B')$. This means that $f(x) \in B \cap B'$, so $f(x) \in B$ and $f(x) \in B'$. But then $x \in f^{-1}(B)$ and $x \in f^{-1}(B')$, i. e. $x \in f^{-1}(B) \cap f^{-1}(B')$.
Conversely, if $x \in f^{-1}(B) \cap f^{-1}(B')$, then $f(x) \in B$ and $f(x) \in B'$. That is, $f(x) \in B \cap B'$ which means that $x \in f^{-1}(B \cap B')$.

Exercise 1.3. For each of the following statements of the form "If P then Q", write its inverse, converse, and contrapositive statements. State which versions of the statement are true, and conclude whether P and Q are equivalent or not.

(i) If a function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is continuous at a, then it is differentiable at a.

(ii) If $x < 0$, then $x^2 - x > 0$.

Solution.

(i) This is false; consider the absolute value function $x \longmapsto |x|$ at 0.
- The inverse is: If a function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is not continuous at a, then it is not differentiable at a. This is true.
- The converse is: If a function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is differentiable at a, then it is continuous at a. This is true.
- The contrapositive is: If a function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is not differentiable at a, then it is not continuous at a. This is false.

Consequently, this is not an equivalence.

(ii) This is true.
- The inverse is: If $x \geq 0$, then $x^2 - x \leq 0$. This is not true. In fact, it fails for any sufficiently large x, say $x > 1$.
- The converse is: If $x^2 - x > 0$, then $x < 0$. This is false.
- The contrapositive is: If $x^2 - x \leq 0$, then $x \geq 0$. This is true.

Consequently, this is not an equivalence either.

Exercise 1.4. Write the negation of the following sentence (note: be careful when negating statements involving there exists and for all!). State whether either statement and its negation are true.

(i) For every number x, there is at least one number y that $e^y = x$. Or, more formally, "for all $x \in \mathbb{R}$, there exists $y \in \mathbb{R}$ such that $e^y = x$."

(ii) There is at least one number a in $[0, 1]$ such that the derivative of x^3 at $x = a$ is 0. Or, more formally, "There exists $a \in [0, 1]$ such that the derivative of x^3 at $x = a$ is 0."

(iii) For every function $f : [0, 1] \longrightarrow [0, 1]$ which is continuous and strictly increasing on $[0, 1]$, there is a unique function $g : [0, 1] \longrightarrow [0, 1]$ such that $g(f(x)) = x$ for every $x \in [0, 1]$.

Solution.

(i) The negation is: There is some number x such that for all numbers y we have $e^y \neq x$. The statement is false and its negation is true: for $x < 0$ there can be no y with $e^y = x$.

(ii) The negation is: For all numbers $a \in [0, 1]$ the derivative of x^3 at $x = a$ is nonzero. The statement is true and its negation is false: the derivative of x^3 at $x = 0$ is 0.

(iii) The negation is: There is a continuous and strictly increasing function $f : [0, 1] \longrightarrow [0, 1]$ such that either for every function $g : [0, 1] \longrightarrow [0, 1]$ we have $g(f(x)) \neq x$ for some $x \in [0, 1]$ or there are at least two functions $g : [0, 1] \longrightarrow [0, 1]$ such that $g(f(x)) = x$ for all $x \in [0, 1]$.

Consider the continuous and strictly increasing function $f : [0, 1] \longrightarrow [0, 1]$ with $f(x) = x/2$ and the functions $g_1, g_2 : [0, 1] \longrightarrow [0, 1]$ with

$$g_1(x) = \begin{cases} 2x & \text{if } x \leq 1/2 \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad g_2(x) = \begin{cases} 2x & \text{if } x \leq 1/2 \\ 1 & \text{otherwise.} \end{cases}$$

Then $g_1(f(x)) = x = g_2(f(x))$ for all $x \in [0, 1]$ but $g_1 \neq g_2$. Consequently the original statement was false and its negation is true.

Exercise 1.5. Solve Munkres §1.3, Exercise 1: Define two points (x_0, y_0) and (x_1, y_1) of the plane to be equivalent if $y_0 - x_0^2 = y_1 - x_1^2$. Check that this is an equivalence relation and describe the equivalence classes.

Solution. For every $(x, y) \in \mathbb{R}^2$ we have of course that $y - x^2 = y - x^2$. So this relation is reflexive. Similarly, if $y_0 - x_0^2 = y_1 - x_1^2$ then surely $y_1 - x_1^2 = y_0 - x_0^2$ and if additionally $y_1 - x_1^2 = y_2 - x_2^2$ then $y_0 - x_0^2 = y_2 - x_2^2$ as well. Hence, the relation is symmetric and transitive. So we have an equivalence relation.

To describe the equivalence classes, let $(x, y) \in \mathbb{R}^2$. The equivalence containing (x, y) is then

$$\{(x', y') : y' - (x')^2 = y - x^2\} = \{(x', y') : y' - (x')^2 = c\}$$

where $c = y - x^2$. Consequently, the equivalence classes are the parabolas $y = x^2 + c$ for $c \in \mathbb{R}$.

Exercise 1.6. For each of the following relations defined on $\mathbb{R}$, the set of all real numbers, check whether the relation is reflexive, symmetric, and transitive (prove the property is true, or find a counterexample). Conclude whether each of the relations is an equivalence relation.

(i) $a \sim b$ means $ab \neq 0$.
(ii) $a \sim b$ means $ab \geq 0$.
(iii) $a \sim b$ means $\lfloor a \rfloor = \lfloor b \rfloor$ (where $\lfloor x \rfloor$ denotes the largest integer less than or equal to x, the "floor" of x).

Solution.

(i) For $a = 0$ we have $a \cdot a = 0$. So this relation is not reflexive. But if $a \sim b$, i. e. $ab \neq 0$, then certainly $ba = ab \neq 0$, that is, $b \sim a$. Hence, $\sim$ is symmetric. If $a \sim b$ and $b \sim c$, then $ab \neq 0$ and $bc \neq 0$. This implies that $a \neq 0$, $b \neq 0$ and $c \neq 0$, so $ac \neq 0$ as well. So $\sim$ is transitive. In summary, $\sim$, not being reflexive, is not an equivalence relation.
(ii) For any real number a, we have $a^2 \geq 0$. So $\sim$ is reflexive. If $a \sim b$, then $ab \geq 0$ and therefore certainly $ba = ab \geq 0$, i. e. $b \sim a$. So $\sim$ is also symmetric. But $-1 \sim 0$ and $0 \sim 1$ while $-1 \nsim 1$ because $(-1) \cdot 1 = -1 \ngeq 0$. Hence, $\sim$ is not transitive and therefore not an equivalence relation either.
(iii) Certainly $\lfloor a \rfloor = \lfloor a \rfloor$ for every $a \in \mathbb{R}$ which means that $\sim$ is reflexive. Also, if $\lfloor a \rfloor = \lfloor b \rfloor$, then we also have $\lfloor b \rfloor = \lfloor a \rfloor$ and therefore $\sim$ is also symmetric. If $a \sim b$ and $b \sim c$, then $\lfloor a \rfloor = \lfloor b \rfloor = \lfloor c \rfloor$ and therefore $a \sim c$ as well. Consequently, $\sim$ is an equivalence relation.

Exercise 1.7.

(i) Let A, B be sets. Show that there is a bijection between $A \times B$ and $B \times A$.
(ii) Let A and B be sets. In class, we defined the *mapping set*

$$\mathrm{Maps}(A, B) = \{f : A \longrightarrow B\}$$

to be the set whose elements are distinct functions $f : A \longrightarrow B$ (two functions f_1 and f_2 are equal if $f_1(a) = f_2(a)$ for every $a \in A$).

Let X be a set. In class, we defined, for a natural number n, the product X^n to be the set of ordered n−tuples of elements of X:

$$X^n = \{(x_1, \ldots, x_n) | x_i \in X\}.$$

Prove that there is a natural bijection

$$X^n \cong \mathrm{Maps}(\{1, \ldots, n\}, X).$$

In fact, Munkres §1.5 simply defines X^n to be $\mathrm{Maps}(\{1, \ldots, n\}, X)$!
(iii) In light of the previous part, let's define

$$\text{“}X^\infty\text{”} = \mathrm{Maps}(\mathbb{N}, X).$$

Equivalently (as in part (ii)), "X^∞" can be described as the set of infinite-length tuples $(x_1, x_2, \ldots)$. In fact, the notation "X^∞" is a little imprecise—as there are many non-equivalent infinite-sized sets (for instance, $\mathbb{R}$ and $\mathbb{N}$). In constrast, all sets with n elements are equivalent[1]. So, instead, we will use the notation

$$X^\omega = \text{“}X^\infty\text{”} = \mathrm{Maps}(\mathbb{N}, X)$$

(where ω, the Greek letter "omega", is typically used to indicate the "size", or *cardinality*, of $\mathbb{N}$).

If X is a non-empty set, and n any natural number, find a bijective map $f : X^n \times X^\omega \longrightarrow X^\omega$.

Solution.

[1] Two sets A and B are *equivalent* if one can find a bijection $A \xrightarrow{\sim} B$.

(i) Define a function $f\colon A\times B \longrightarrow B\times A$ by setting $f((a,b)) = (b,a)$ and $g\colon B\times A \longrightarrow A\times B$ by $g((b,a)) = (a,b)$. Observe that we have $g(f((a,b))) = (a,b)$ and $f(g((b,a))) = (b,a)$ for all $a\in A$ and $b\in B$. This means that f admits an inverse function and therefore must be a bijection.

(ii) Define a function $f\colon \mathrm{Maps}(\{1,\dots,n\},X) \longrightarrow X^n$ by setting $f(g) = (g(1),\dots,g(n))$ whenever $g\colon\{1,\dots,n\}\longrightarrow X$ is a function. Conversely, define a function $h\colon X^n \longrightarrow \mathrm{Maps}(\{1,\dots,n\},X)$ as follows: Let $x=(x_1,\dots,x_n)\in X^n$. Then there is a function $\widetilde{x}\colon\{1,\dots,n\}\longrightarrow X$ defined by $\widetilde{x}(i)=x_i$ and we set $h(x)=\widetilde{x}$. The functions f and h are inverses of each other: Compute

$$f(h(x)) = f(\widetilde{x}) = (\widetilde{x}(1),\dots,\widetilde{x}(n)) = (x_1,\dots,x_n) = x$$
$$h(f(g))(i) = h((g(1),\dots,g(n)))(i) = g(i), \quad \text{for all } i\in\{1,\dots,n\}$$

where $x=(x_1,\dots,x_n)\in X^n$ and $g\colon\{1,\dots,n\}\longrightarrow X$. Since f admits an inverse function h, it must be a bijection $\mathrm{Maps}(\{1,\dots,n\},X)\xrightarrow{\sim} X^n$.

(iii) Define $f\colon X^n\times X^\omega \longrightarrow X^\omega$ by setting

$$f((x_1,\dots,x_n),(y_1,\dots)) = (x_1,\dots,x_n,y_1,\dots)$$

for $(x_1,\dots,x_n)\in X^n$ and $(y_1,\dots)\in X^\omega$. Again, we can define a function $h\colon X^\omega\longrightarrow X^n\times X^\omega$ in the other direction by

$$h(x_1,\dots) = ((x_1,\dots,x_n),(x_{n+1},\dots)).$$

Then f and g are inverses of each other: Compute

$$f(h(x_1,\dots)) = f((x_1,\dots,x_n),(x_{n+1},\dots)) = (x_1,\dots,x_n,x_{n+1},\dots)$$
$$h(f((y_1,\dots,y_n),(x_1,\dots))) = h(y_1,\dots,y_n,x_1,\dots) = ((y_1,\dots,y_n),(x_1,\dots))$$

for $(x_1,\dots)\in X^\omega$ and $(y_1,\dots,y_n)\in X^n$. Consequently, f must be a bijection $X^n\times X^\omega\xrightarrow{\sim} X^\omega$.