Math 440 Homework 2

Due Friday, Sept. 8, 2017 by 4 pm

Please remember to write down your name on your assignment.

Please submit your homework to our TA Viktor Kleen, either in his mailbox (in KAP 405) or under the door of his office (KAP 413). You may also e-mail your solutions to Viktor provided:

- you have typed your homework solutions; or
- you are able to produce a very high quality scanned PDF (no photos please!),

1. Give a careful proof of De Morgan's identities: for $X, \{A_i\}_{i\in I}$ sets, the following two equalities hold:
$$X - \bigcup_{i\in I} A_i = \bigcap_{i\in I}(X - A_i)$$
$$X - \bigcap_{i\in I} A_i = \bigcup_{i\in I}(X - A_i).$$

2. Let X be any set. Show that there is a bijection of the power set of X (the set of subsets of X)
$$\mathcal{P}(X) = \{A|A \subset X\}$$
and the set of maps from X to $\{0, 1\}$,
$$Maps(X, \{0, 1\}).$$
(in other words, construct a map in either direction and either argue that it is bijective or construct an inverse (and prove that it is an inverse)).

3. Let X be any set. Complete the proof begun in class that the *discrete metric* on X $d_{discrete} : X \times X \to [0, +\infty)$, defined as
$$d_{discrete}(x, y) = \begin{cases} 0 & x = y \\ 1 & x \neq y \end{cases}$$
is indeed a metric. (**Note**: We proved properties (i) and (ii) of being a metric; this question is specifically asking you to prove the triangle inequality, property (iii)).

   Then, show that every subset $U \subset X$ is an open set with respect to the discrete metric.

4. (a) Consider $\mathbb{R}^n$ with its *Euclidean metric*
$$d_{Eu}(\mathbf{x}, \mathbf{y}) = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2},$$
and its *taxi-cab metric*
$$d_{Ta}(\mathbf{x}, \mathbf{y}) = |x_1 - y_1| + |x_2 - y_2| + \cdots + |x_n - y_n|.$$
Show that there exists positive constants c_1, c_2 such that, for any points $x, y \in \mathbb{R}^n$.
$$c_1 d_{Eu}(\mathbf{x}, \mathbf{y}) \leq d_{Ta}(\mathbf{x}, \mathbf{y}) \leq c_2 d_{Eu}(\mathbf{x}, \mathbf{y})$$

(c_1 and c_2 are *not* allowed to depend on $\mathbf{x}$ and $\mathbf{y}$ above).

Using the above fact (which you may want to call a Lemma), prove that a subset $U \subset \mathbb{R}^n$ is open with respect to the d_{Eu} metric if and only if U is open with respect to the d_{Ta} metric. In other words, d_{Eu} and d_{Ta} have the same open sets, or *induce the same topology.*[1]

(b) More generally, given an integer $p \geq 1$, define

$$d_p(\mathbf{x}, \mathbf{y}) := \left[\sum_{i=1}^{n} |x_i - y_i|^p\right]^{1/p}.$$

You may assume that d_p is a metric (in particular, that it satisfies axioms (i), (ii) and (iii)). Show using similar methods that d_p also induces the same topology as d_{Eu}.

5. (a) Let $x \in \mathbb{Q}$ be a rational number. In class, we defined the *2-adic norm*

$$\begin{cases} \text{If } x \neq 0, |x|_2 = 2^{-n} \text{ where } n \in \mathbb{Z} \text{ is the unique integer such that } x = 2^n \frac{p}{q} \text{ with } p,\ q \text{ both odd} \\ \text{If } x = 0, |x|_2 = 0. \end{cases}$$

and the *2-adic metric*

$$d_2 : \mathbb{Q} \times \mathbb{Q} \to [0, \infty)$$

by

$$d_2(x, y) = |x - y|_2.$$

(**Note (9/6/2017)**: observe that there is a sign change (n vs. $-n$): if $x - y = 2^n \frac{p}{q}$ where p and q are odd, then $d_2(x, y) = 2^{-n}$.)

Prove that $(\mathbb{Q}, d_2)$ is a metric space. In fact, prove that d_2 satisfies conditions (i) or (ii) of being a metric, as well as a condition *stronger* than (iii):

$$\text{For all } x,\ y,\ z \in \mathbb{Q},\ d_2(x, z) \leq \max(d_2(x, y), d_2(y, z)). \tag{1}$$

(note that $\max(d_2(x, y), d_2(y, z)) \leq d_2(x, y) + d_2(y, z)$, so this condition indeed implies the triangle inequality).

Remark: Metrics satisfying this stronger condition (1) are sometimes called *ultra-metrics* or *non-Archimedean metrics*.

(b) Prove that for any $x \in \mathbb{Q}$, there is some $r > 0$ such that the complement $\mathbb{Q} - B_{d_2}(x, r)$ is an open set[2].

(in contrast, note that in the Euclidean metric $\mathbb{R} - (x - r, x + r) = (-\infty, r] \cup [r, +\infty)$ is *never* open).

[1] We say a pair of metrics d_1, d_2 *induce the same topology* on a set X if they have the same open sets, meaning for any subset $U \subset X$, U is open with respect to d_1 iff it is open with respect to d_2.

[2] Recall that $B_d(x, r)$ is the *ball of radius r centered at x in the metric d*, which we defined in class. In short, $B_d(x, r) = \{y \in X | d(x, y) < r\}$.

6. **Product metrics**. If (M_1, d_1) and (M_2, d_2) are metric spaces, then one can define a distance function d on the Cartesian product $M_1 \times M_2$ by

$$d((m,n),(m',n')) = d_1(m,m') + d_2(n,n'). \tag{2}$$

(a) Show that d defines a metric on $M_1 \times M_2$, called the *(standard) product metric.*

(b) Prove that if $U_1 \subset M_1$ is open and $U_2 \subset M_2$ is open, then $U_1 \times U_2 \subset M_1 \times M_2$ is open. Conversely, is it true that every open set $V \subset M_1 \times M_2$ is of the form $U_1 \times U_2$ for some U_1, U_2? (prove or find a counterexample, with justification).