Homework 2

Exercise 2.1. Give a careful proof of De Morgan's identities: for X, $\{A_i\}_{i\in I}$ sets, the following two equalities hold:

$$X \setminus \bigcup_{i\in I} A_i = \bigcap_{i\in I}(X \setminus A_i)$$
$$X \setminus \bigcap_{i\in I} A_i = \bigcup_{i\in I}(X \setminus A_i).$$

Solution. Suppose $x \in X \setminus \bigcup_{i\in I} A_i$. This means that $x \in X$ but $x \notin \bigcup_{i\in I} A_i$. The latter means that $x \notin A_i$ for every $i \in I$. We conclude that $x \in X \setminus A_i$ for every $i \in I$, i. e. $x \in \bigcap_{i\in I}(X \setminus A_i)$.

Conversely, if $x \in \bigcap_{i\in I}(X \setminus A_i)$, then this means that $x \in X \setminus A_i$ for every $i \in I$. That is, $x \in X$ and $x \notin A_i$ for all $i \in I$. The latter is equivalent to $x \notin \bigcup_{i\in I} A_i$. So $x \in X \setminus \bigcup_{i\in I} A_i$.

Similarly, suppose $x \in X \setminus \bigcap_{i\in I} A_i$. Then $x \in X$ but $x \notin \bigcap_{i\in I} A_i$. So there is some $j \in I$ such that $x \notin A_j$. For this j we then have $x \in X \setminus A_j \subset \bigcup_{i\in I}(X \setminus A_i)$.

Conversely, if $x \in \bigcup_{i\in I}(X \setminus A_i)$, then there is some $j \in I$ such that $x \in X \setminus A_j$, i. e. $x \in X$ but $x \notin A_j$. This implies that $x \notin \bigcap_{i\in I} A_i$ because $A_j \supset \bigcap_{i\in I} A_i$. Hence, $x \in X \setminus \bigcap_{i\in I} A_i$.

Exercise 2.2. Let X be any set. Show that there is a bijection of the power set of X (the set of subsets of X)

$$\mathscr{P}(X) = \{A : A \subset X\}$$

and the set of maps from X to $\{0, 1\}$,

$$\mathrm{Maps}(X, \{0, 1\}).$$

Solution. Define a map $f\colon \mathrm{Maps}(X, \{0, 1\}) \longrightarrow \mathscr{P}(X)$ by setting $f(\chi) = \{x \in X : \chi(x) = 1\}$ for a function $\chi \in \mathrm{Maps}(X, \{0, 1\})$. Conversely, given a subset $A \subset X$ define a function $\chi_A\colon X \longrightarrow \{0, 1\}$ by

$$\chi_A(x) = \begin{cases} 0 & \text{if } x \notin A \\ 1 & \text{if } x \in A. \end{cases}$$

Define $g\colon \mathscr{P}(X) \longrightarrow \mathrm{Maps}(X, \{0, 1\})$ by $g(A) = \chi_A$. We check that f and g are inverse functions:

$$f(g(A)) = f(\chi_A) = \{x \in X : \chi_A(x) = 1\} = \{x \in X : x \in A\} = A$$
$$g(f(\chi))(x) = \begin{cases} 0 & \text{if } x \notin f(\chi), \text{ i. e. } \chi(x) = 0 \\ 1 & \text{if } x \in f(\chi), \text{ i. e. } \chi(x) = 1, \end{cases}$$

for $A \subset X$ and $\chi \in \mathrm{Maps}(X, \{0, 1\})$. We conclude that f and g are inverse bijections $\mathrm{Maps}(X, \{0, 1\}) \cong \mathscr{P}(X)$.

Exercise 2.3. Let X be a set. Complete the proof that the *discrete metric* $d_{\text{discrete}}\colon X \times X \longrightarrow [0. + \infty)$ on X, defined as

$$d_{\text{discrete}}(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y, \end{cases}$$

is indeed a metric.

Then, show that every subset $U \subset X$ is an open set with respect to the discrete metric.

Solution. We prove the triangle inequality for d_{discrete}. So suppose, $x, y, z \in X$. There are three cases: $x = y = z$, or exactly two points of x, y, z are equal, or x, y and z are distinct points. In the first case, we have

$$0 = d_{\text{discrete}}(x, y) + d_{\text{discrete}}(y, z) \geq d_{\text{discrete}}(x, z) = 0.$$

In the second case, if say $x = y$ but $y \neq z$, we have

$$1 = d_{\text{discrete}}(x, y) + d_{\text{discrete}}(y, z) \geq d_{\text{discrete}}(x, z) = 1$$

and if $x \neq y$ but $x = z$, then

$$1 = d_{\text{discrete}}(x, y) + d_{\text{discrete}}(y, z) \geq d_{\text{discrete}}(x, z) = 0.$$

Finally, in the third case, we have

$$2 = d_{\text{discrete}}(x, y) + d_{\text{discrete}}(y, z) \geq d_{\text{discrete}}(x, z) = 1.$$

Now, let $U \subset X$ be a subset and let $x \in U$. Observe that $B_{d_{\text{discrete}}}(x, 1/2) = \{x\} \subset U$. So U is a neighborhood of x. But since x was arbitrary we conclude that U is a neighborhood of each of its points, that is, U is open.

Exercise 2.4.

(i) Consider $\mathbb{R}^n$ with its *Euclidean metric*

$$d_{\text{Eu}}(\mathbf{x}, \mathbf{y}) = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2}$$

and its *taxi-cab metric*

$$d_{\text{Ta}}(\mathbf{x}, \mathbf{y}) = |x_1 - y_1| + |x_2 - y_2| + \cdots + |x_n - y_n|.$$

Show that there exist positive constants c_1, c_2 such that, for any points $x, y \in \mathbb{R}^n$,

$$c_1 d_{\text{Eu}}(\mathbf{x}, \mathbf{y}) \leq d_{\text{Ta}}(\mathbf{x}, \mathbf{y}) \leq c_2 d_{\text{Eu}}(\mathbf{x}, \mathbf{y}).$$

Note that c_1 and c_2 are *not* allowed to depend on $\mathbf{x}$ and $\mathbf{y}$ above.

Using the above fact (which you may want to call a Lemma), prove that a subset $U \subset \mathbb{R}^n$ is open with respect to the d_{Eu} metric if and only if U is open with respect to the d_{Ta} metric. In other words, d_{Eu} and d_{Ta} have the same open sets, or *induce the same topology.*[1]

(ii) More generally, given an integer $p \geq 1$, define

$$d_p(\mathbf{x}, \mathbf{y}) := \left[\sum_{i=1}^{n} |x_i - y_i|^p\right]^{1/p}$$

You may assume that d_p is a metric. Show using similar methods that d_p also induces the same topology as d_{Eu}.

Solution.

(i) First, let's assume that d_{Eu} and d_{Ta} are in fact equivalent and using this show that they induce the same topology on $\mathbb{R}^n$. For this, assume that $U \subset \mathbb{R}^n$ is open with respect to d_{Eu} and let $x \in U$. There is some $\varepsilon > 0$ such that $B_{d_{\text{Eu}}}(x, \varepsilon) \subset U$. Now, suppose that $y \in B_{d_{\text{Ta}}}(x, c_1\varepsilon)$. Then $d_{\text{Eu}}(x, y) \leq 1/c_1 \cdot d_{\text{Ta}}(x, y) < \varepsilon$ and therefore $y \in B_{d_{\text{Eu}}}(x, \varepsilon) \subset U$. So, $B_{d_{\text{Ta}}}(x, c_1\varepsilon) \subset U$. We conclude that U is open with respect to d_{Ta} as well.

Conversely, if $U \subset \mathbb{R}^n$ is open with respect to d_{Ta} and $x \in U$, let $\varepsilon > 0$ be small enough such that $B_{d_{\text{Ta}}}(x, \varepsilon) \subset U$. Suppose $y \in B_{d_{\text{Eu}}}(x, \varepsilon/c_2)$. Then $d_{\text{Ta}}(x, y) \leq c_2 d_{\text{Eu}}(x, y) < \varepsilon$ and therefore $y \in B_{d_{\text{Ta}}}(x, \varepsilon)$. Hence, $B_{d_{\text{Eu}}}(x, \varepsilon/c_2) \subset U$ and we conclude that U is open with respect to d_{Eu} as well.

[1]We say a pair of metric d_1, d_2 *induce the same topology* on a set X if they have the same open sets, meaning for any subset $U \subset X$, U is open with respect to d_1 if and only if it is open with respect to d_2.

To prove that d_{Eu} and d_{Ta} are in fact equivalent, note that for $a, b \in \mathbb{R}$ we have $0 \leq (a-b)^2 = a^2 - 2ab + b^2$ and therefore $2ab \leq a^2 + b^2$. Then, using that $2|x_i - y_i||x_j - y_j| \geq 0$ for all $1 \leq i, j \leq n$, we can compute

$$\begin{aligned}
(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2 &\leq (x_1 - y_1)^2 + \cdots + (x_n - y_n)^2 + \sum_{\substack{i,j=1 \\ i<j}}^{n} 2|x_i - y_i||x_j - y_j| = \\
&= [|x_1 - y_1| + \cdots + |x_n + y_n|]^2 = d_{\text{Ta}}(\mathbf{x}, \mathbf{y})^2 = \\
&= (x_1 - y_1)^2 + \cdots + (x_n - y_n)^2 + \sum_{\substack{i,j=1 \\ i<j}}^{n} 2|x_i - y_i||x_j - y_j| \leq \\
&\leq (x_1 - y_1)^2 + \cdots + (x_n - y_n)^2 + \sum_{\substack{i,j=1 \\ i<j}}^{n} \left[(x_i - y_i)^2 + (x_j - y_j)^2\right] \leq \\
&\leq n^2 \left[(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2\right].
\end{aligned}$$

Taking square roots this implies $d_{\text{Eu}}(\mathbf{x}, \mathbf{y}) \leq d_{\text{Ta}}(\mathbf{x}, \mathbf{y}) \leq n\, d_{\text{Eu}}(\mathbf{x}, \mathbf{y})$.

(ii) For $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ let $z_{\max} = \max_i |x_i - y_i|$. Compute:

$$\begin{aligned}
\sum_{i=1}^{n} |x_i - y_i|^p &\leq n\, z_{\max}^p \leq n \left[\sum_{i=1}^{n} |x_i - y_i|\right]^p \\
\sum_{i=1}^{n} |x_i - y_i|^p &\geq z_{\max}^p \geq \left[\frac{1}{n} \sum_{i=1}^{n} |x_i - y_i|\right]^p
\end{aligned}$$

Combining these and taking p^{th} roots, we conclude that $n^{-1}\, d_{\text{Ta}}(\mathbf{x}, \mathbf{y}) \leq d_p(\mathbf{x}, \mathbf{y}) \leq n^{1/p}\, d_{\text{Ta}}(\mathbf{x}, \mathbf{y})$. So d_p and d_{Ta} are equivalent metrics and by the same argument as in part (i) they induce the same topologies on $\mathbb{R}^n$. Hence, d_p also induces the same topology as d_{Eu}.

Exercise 2.5.

(i) Let $x \in \mathbb{Q}$ be a rational number. In class, we defined the $2-$*adic norm*

$$|x|_2 = \begin{cases} 2^{-n} & \text{if } x \neq 0 \text{ and } n \in \mathbb{Z} \text{ is the unique integer such that } x = 2^n \frac{p}{q} \text{ with } p \text{ and } q \text{ odd} \\ 0 & \text{if } x = 0 \end{cases}$$

and the $2-$*adic metric*

$$d_2 : \mathbb{Q} \times \mathbb{Q} \longrightarrow [0, \infty)$$

by $d_2(x, y) = |x - y|_2$. Prove that $(\mathbb{Q}, d_2)$ is a metric space. In fact, prove that d_2 satisfies conditions (i) and (ii) of being a metric, as well as a condition *stronger* than (iii):

$$\text{For all } x, y, z \in \mathbb{Q},\ d_2(x, z) \leq \max(d_2(x, y), d_2(y, z)).$$

Metrics satisfying this stronger condition are sometimes called *ultrametrics* or *non-Archimedean metrics.*

(ii) Prove that for any $x \in \mathbb{Q}$, there is some $r > 0$ such that the complement $\mathbb{Q} \setminus B_{d_2}(x, r)$ is an open set.[2] In contrast, note that in the Euclidean metric $\mathbb{R} \setminus (x - r, x + r)$ is *never* open.

Solution.

[2]Recall that $B_d(x, r)$ is the *ball of radius r centered at x in the metric d*, which we defined in class. In short, we defined $B_d(x, r) = \{y \in X : d(x, y) < r\}$.

(i) First note that $|x-y|_2 = 0$ if and only if $x = y$ because $2^{-n} \neq 0$ for all $n \in \mathbb{Z}$. Also, if $|x-y|_2 = 2^{-n}$, then $x - y = 2^n \frac{p}{q}$ for some odd p and q. So, $y - x = 2^n \frac{-p}{q}$ and therefore $|y-x|_2 = 2^{-n} = |x-y|_2$.
Now, suppose $a, b \in \mathbb{Q}$ are nonzero rational numbers with $|a|_2 = 2^{-n}$ and $|b|_2 = 2^{-m}$. This means that $a = 2^n \frac{p}{q}$ and $b = 2^m \frac{p'}{q'}$ for odd integers $p, p', q, q' \in \mathbb{Z}$. Assume without loss of generality that $|a|_2 \geq |b|_2$. Then $m \geq n$ and therefore

$$a + b = 2^n \frac{p}{q} + 2^m \frac{p'}{q'} = 2^n \left[\frac{p}{q} + 2^{m-n} \frac{p'}{q'} \right] = 2^n \frac{pq' + 2^{m-n} p'q}{qq'}$$

Because q and q' are both odd, the denominator qq' will be odd as well. Because of our assumption that $m - n \geq 0$ the numerator $pq' + 2^{m-n} p'q$ is an integer. It follows that $pq' + 2^{m-n} p'q = 2^\ell r$ for $\ell \geq 0$ and $r \in \mathbb{Z}$ odd. Consequently, $a + b = 2^{n+\ell} \frac{r}{qq'}$ and $|a+b|_2 = 2^{-n-\ell} \leq 2^{-n} = \max\{|a|_2, |b|_2\}$.
We conclude that

$$d_2(x,z) = |x-z|_2 = |x-y+y-z|_2 \leq \max\{|x-y|_2, |y-z|_2\} = \max\{d_2(x,y), d_2(y,z)\}$$

for $x, y, z \in \mathbb{Q}$.

(ii) Let $x \in \mathbb{Q}$ and suppose $r > 0$. Suppose $y \in \mathbb{Q} \setminus B_{d_2}(x,r)$ and $z \in B_{d_2}(y,r)$. The ultrametric triangle inequality from part (i) implies that

$$r \leq |x-y|_2 = |x-z-y+z|_2 \leq \max\{|x-z|_2, |y-z|_2\}.$$

For sake of contradiction, suppose that $|x-z|_2 \leq |y-z|_2$. Then we would have $r \leq |x-y|_2 \leq |y-z|_2 < r$ which is impossible. So it must be the case that $|y-z|_2 \leq |x-z|_2$. But then $r \leq |x-y|_2 \leq |x-z|_2$ and therefore $z \in \mathbb{Q} \setminus B(x,r)$. Since z was arbitrary we conclude that $B_{d_2}(y,r) \subset \mathbb{Q} \setminus B_{d_2}(x,r)$ and, because $y \in \mathbb{Q} \setminus B_{d_2}(x,r)$ was arbitrary as well, it follows that $\mathbb{Q} \setminus B_{d_2}(x,r)$ is open.

Exercise 2.6. If (M_1, d_1) and (M_2, d_2) are metric spaces, then one can define a distance function d on the Cartesian product $M_1 \times M_2$ by

$$d((m,n),(m',n')) = d_1(m,m') + d_2(n,n').$$

(i) Show that d defines a metric on $M_1 \times M_2$, called the *(standard) product metric*.
(ii) Prove that if $U_1 \subset M_1$ is open and $U_2 \subset M_2$ is open, then $U_1 \times U_2 \subset M_1 \times M_2$ is open. Conversely, is it true that every open set $V \subset M_1 \times M_2$ is of the form $U_1 \times U_2$ for some U_1, U_2?

Solution.

(i) Because $d_1(m,m')$ and $d_2(n.n')$ are both always non-negative, $d((m,n),(m',n')) = 0$ is equivalent to $d_1(m,m') = 0$ and $d_2(n,n') = 0$. But this is equivalent to $(m,n) = (m',n')$. Also,

$$d((m,n),(m',n')) = d_1(m,m') + d_2(n,n') = d_1(m',m) + d_2(n',n) = d((m',n'),(m,n)).$$

Finally, suppose that $(m,n), (m',n'), (m'',n'') \in M_1 \times M_2$. Then by the triangle inequalities for d_1 and d_2 we conclude

$$\begin{aligned} d((m,n),(m'',n'')) &= d_1(m,m'') + d_2(n,n'') \leq \\ &\leq d_1(m,m') + d_1(m',m'') + d_2(n,n') + d_2(n',n'') = \\ &= d((m,n),(m',n')) + d((m',n'),(m'',n'')). \end{aligned}$$

(ii) Suppose that $U_1 \subset M_1$ and $U_2 \subset M_2$ are both open and let $(m,n) \in U_1 \times U_2$. Let $\varepsilon_1 > 0$ be small enough such that $B_{d_1}(m, \varepsilon_1) \subset U_1$ and similarly let $\varepsilon_2 > 0$ be small enough such that $B_{d_2}(n, \varepsilon_2) \subset U_2$. Pick any $\varepsilon > 0$ with $\varepsilon < \min\{\varepsilon_1, \varepsilon_2\}$ and suppose $(x,y) \in B_d((m,n), \varepsilon)$. Then

$$d_1(m,x) \leq d_1(m,x) + d_2(n,y) = d((m,n),(x,y)) < \varepsilon < \varepsilon_1$$
$$d_2(n,y) \leq d_1(m,x) + d_2(n,y) = d((m,n),(x,y)) < \varepsilon < \varepsilon_2$$

and therefore $x \in U_1$ and $y \in U_2$. So, $(x, y) \in U_1 \times U_2$ and we conclude that $B_d((m, n), \varepsilon) \subset U_1 \times U_2$. This implies that $U_1 \times U_2$ is open since $(m, n) \in U_1 \times U_2$ was arbitrary.
The converse isn't true. For example $(0, 1) \times (0, 1) \cup (2, 3) \times (2, 3) \subset \mathbb{R} \times \mathbb{R}$ is open but not a cartesian product of open sets $U_1, U_2 \subset \mathbb{R}$.