Math 440 Homework 3

Due Friday, Sept. 15, 2017 by 4 pm

Please remember to write down your name on your assignment.

Please submit your homework to our TA Viktor Kleen, either in his mailbox (in KAP 405) or under the door of his office (KAP 413). You may also e-mail your solutions to Viktor provided:

- you have typed your homework solutions; or
- you are able to produce a very high quality scanned PDF (no photos please!),

1. Prove or disprove the following property: if a metric space (X, d) has at least two elements, then it admits an open subset which is neither X nor the empty set $\emptyset$.

2. Let $X = (X, d_X)$, $Y = (Y, d_Y)$, $Z = (Z, d_Z)$ and $W = (W, d_W)$. Prove that if $f : X \to Z$ and $g : Y \to W$ are each continuous, then
$$f \times g : X \times Y \to Z \times W$$
sending $(x, y) \mapsto (f(x), g(y))$, is continuous, with respect to the product metrics on $X \times Y$ and $Z \times W$ we studied on HW2 # 6.

3. The goal of this exercise is to prove continuity of some of the standard algebraic operations on $\mathbb{R}$, thought of as maps $\mathbb{R}^2 \to \mathbb{R}$; we will focus on addition
$$\begin{aligned} + : \mathbb{R}^2 &\to \mathbb{R} \\ (x, y) &\mapsto x + y \end{aligned}$$
and multiplication
$$\begin{aligned} \cdot : \mathbb{R}^2 &\to \mathbb{R} \\ (x, y) &\mapsto x \cdot y. \end{aligned}$$
(*Note*: By similar techniques, one can prove that *multiplication by a scalar* $c \cdot - : \mathbb{R} \to \mathbb{R}$, and the *reciprocal map*: $\mathbb{R} - \{0\} \to \mathbb{R}$ are continuous; by composition, one obtains that for example, $(x, y) \mapsto ax + by$ is continuous for any a, b and $(x, y) \mapsto x/y$ (for $y \neq 0$) is continuous).

   Use the standard Euclidean metric $d(x, y) = |x - y|$ on $\mathbb{R}$ and the 'taxicab' metric on $\mathbb{R}^2$ given by
$$d_{Ta}((x_1, x_2), (y_1, y_2)) = |x_1 - y_1| + |x_2 - y_2|.$$
(this metric is equivalent to the usual Euclidean metric by your previous homework, so anything you prove about these operations being continuous for d_{Ta} holds for d_{Eu} as well[1]).
   (a) Show that addition is continuous. **Hint**: Given any point (x_0, y_0) you'd like to verify continuity at, and any $\epsilon > 0$, check that $\delta = \epsilon$ works, by noting that $d(x+y, x_0+y_0) \leq$

[1] Why?

$|x - x_0| + |y - y_0|$.

(b) Show that multiplication is continuous. **Hint**: Given any point (x_0, y_0) and $\epsilon > 0$, show that

$$\delta = \min(\frac{\epsilon}{3(|x_0| + |y_0| + 1)}, 1)$$

works. It will be helpful to establish the following inequality:

$$\begin{aligned} d_{\mathbb{R}}(xy, x_0y_0) &\leq |x_0||y - y_0| + |y_0||x - x_0| + |x - x_0||y - y_0| \\ &\leq |x_0| d_{Ta}((x, y), (x_0, y_0)) + |y_0| d_{Ta}((x, y), (x_0, y_0)) + d_{Ta}((x, y), (x_0, y_0))^2. \end{aligned}$$

(where in the second line we've repeatedly applied the fact that $|x-x_0| \leq d_{Ta}((x, y), (x_0, y_0))$ and $|y - y_0| \leq d_{Ta}((x, y), (x_0, y_0))$).

(c) Conclude a result stated in class (without any more ϵ-δ work!): that if $f, g : X \to \mathbb{R}$ are two continuous functions from a metric space to $\mathbb{R}$, then $f + g$ and $f \cdot g$ and continous. *Hint:* use the (i) the fact that compositions of continuous functions are continuous, (ii) problem 2, and (iii) part (a) or (b) respectively. (by similar methods, one can show that f/g when $g \neq 0$, $af + bg$, etc. are continuous).

4. **Constructing open and closed sets**. Let $X := (X, d)$ and $Y := (Y, d')$ be metric spaces and $f : X \to Y$ a map. By a theorem we stated in class on Friday and proved on Monday, f is continuous if and only if the preimage under f of any open (respectively closed set) is open (respectively closed). This suggests an easy way to give many new examples of open and closed sets in a metric space M: write down a function $f : M \to \mathbb{R}$, show that it is continuous (with respect to the ϵ-δ definition, or using other rules of construction continuous functions), and then take the pre-image under f of an open or closed set in $\mathbb{R}$ respectively. (Going further, one can take intersections and/or unions of such sets. . .)

Using this method:

(a) Fix real positive numbers $a_1, \ldots, a_{k+1} > 0$. Show that the generalized ellipsoid $E_{a_1,\ldots,a_{k+1}} := \{(x_1, \ldots, x_{k+1}) | \sum_{i=1}^{k+1} a_i x_i^2 = 1\}$ is a closed subset of $\mathbb{R}^{k+1}$. *Hint*: write this as the preimage of a closed set under a continuous function.

(b) Let $V = \{(x_1, x_2, x_3) | x_1^2 + x_2 < 1 \text{ and } (x_3 - 2)^4 > 4\}$ Show that V is an open subset of $\mathbb{R}^3$ (with its standard Euclidean metric). *Hint*: write this as the intersection of the preimages of two open sets under two continuous functions.