Homework 3

EXERCISE 3.1. Prove or disprove the following property: if a metric space (X, d) has at least two elements, then it admits an open subset which is neither X nor the empty set $\emptyset$.
Solution. Suppose $x \in X$. Because X has at least two elements, the complement $X \setminus \{x\}$ is nonempty. Suppose $y \in X \setminus \{x\}$ and let $r = d(x, y)$. Then certainly $x \notin B_d(y, r)$ because $d(x, y) \not< r$. We conclude that $X \setminus \{x\}$ is open in X, and $X \setminus \{x\} \neq X$ as well as $X \setminus \{x\} \neq \emptyset$.

EXERCISE 3.2. Let $X = (X, d_X)$, $Y = (Y, d_Y)$, $Z = (Z, d_Z)$ and $W = (W, d_W)$ be metric spaces. Prove that if $f: X \longrightarrow Z$ and $g: Y \longrightarrow W$ are each continuous, then

$$f \times g: X \times Y \longrightarrow Z \times W$$

sending $(x, y) \longmapsto (f(x), g(y))$ is continuous, with respect to the product metrics on $X \times Y$ and $Z \times W$.
Solution. Suppose that $(x_0, y_0) \in X \times Y$ and $\varepsilon > 0$. By the continuity of f and g there are $\delta_1, \delta_2 > 0$ such that $f(B_{d_X}(x_0, \delta_1)) \subset B_{d_Z}(f(x_0), \varepsilon/2)$ and $g(B_{d_Y}(y_0, \delta_2)) \subset B_{d_W}(g(y_0), \varepsilon/2)$. Let $\delta = \min\{\delta_1, \delta_2\}$.

Suppose that $(x, y) \in B_{d_{X\times Y}}((x_0, y_0), \delta)$, that is, $d_{X\times Y}((x_0, y_0), (x, y)) < \delta$. This means that we have $d_X(x_0, x) + d_Y(y_0, y) < \delta$. But then $d_X(x_0, x) \leq d_X(x_0, x) + d_Y(y_0, y) < \delta \leq \delta_1$ so that $x \in B_{d_X}(x_0, \delta_1)$, and similarly $y \in B_{d_Y}(y_0, \delta_2)$. By our choice of δ_1 and δ_2 this implies that $d_Z(f(x_0), f(x)) < \varepsilon/2$ and $d_W(g(y_0), g(y)) < \varepsilon/2$. Combining these we find

$$d_{Z\times W}(f \times g(x_0, y_0), f \times g(x, y)) = d_Z(f(x_0), f(x)) + d_W(g(y_0), g(y)) < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

So $f \times g(x, y) \in B_{d_{Z\times W}}(f \times g(x_0, y_0), \varepsilon)$. We conclude that $f \times g$ is continuous with respect to the product metrics.

EXERCISE 3.3. The goal of this exercise is to prove continuity of some of the standard algebraic operations on $\mathbb{R}$, thought of as maps $\mathbb{R}^2 \longrightarrow \mathbb{R}$; we will focus on addition

$$\begin{aligned} +: \mathbb{R}^2 &\longrightarrow \mathbb{R} \\ (x, y) &\longmapsto x + y \end{aligned}$$

and multiplication

$$\begin{aligned} \cdot: \mathbb{R}^2 &\longrightarrow \mathbb{R} \\ (x, y) &\longmapsto xy. \end{aligned}$$

Use the standard Euclidean metric $d(x, y) = |x - y|$ on $\mathbb{R}$ and the "taxicab" metric on $\mathbb{R}^2$ given by

$$d_{\mathrm{Ta}}((x_1, x_2), (y_1, y_2)) = |x_1 - y_1| + |x_2 - y_2|;$$

this metric is equivalent to the usual Euclidean metric, so anything you prove about these operations being continuous for d_{Ta} holds for d_{Eu} as well.

(i) Show that addition is continuous.
(ii) Show that multiplication is continuous.
(iii) Conclude a result stated in class: that if $f, g: X \longrightarrow \mathbb{R}$ are two continuous functions from a metric space to $\mathbb{R}$, then $f + g$ and fg are continuous.

Solution.

(i) Suppose that $(x_0, y_0) \in \mathbb{R}^2$ and $\varepsilon > 0$. Then, if $d((x_0, y_0), (x, y)) = |x_0 - x| + |y_0 - y| < \varepsilon$, we will have

$$d(x_0 + y_0, x + y) = |x_0 + y_0 - x - y| \leq |x_0 - x| + |y_0 - y| < \varepsilon.$$

So we conclude that $+$ is continuous.

(ii) Suppose that $(x_0, y_0) \in \mathbb{R}^2$ and $\varepsilon > 0$. Let $\delta = \min\{\varepsilon/(|x_0| + |y_0| + 1), 1\}$ and suppose that $(x, y) \in \mathbb{R}^2$ satisfies $d((x_0, y_0), (x, y)) = |x_0 - x| + |y_0 - y| < \delta$. Then

$$\begin{aligned} d(x_0y_0, xy) &= |x_0y_0 - xy| = |x_0(y_0 - y) + (x_0 - x)y_0 - (x_0 - x)(y_0 - y)| \leq \\ &\leq |x_0||y_0 - y| + |x_0 - x||y_0| + |x_0 - x||y_0 - y| < \\ &< |x_0|\delta + |y_0|\delta + \delta^2 = \\ &= (|x_0| + |y_0| + \delta)\delta \leq \\ &\leq (|x_0| + |y_0| + 1)\frac{\varepsilon}{|x_0| + |y_0| + 1} = \varepsilon. \end{aligned}$$

Hence, we can conclude that $\cdot : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is continuous.

(iii) First consider the map $\Delta : X \longrightarrow X \times X$ with $\Delta(x) = (x, x)$. We claim that Δ is continuous. To see this, let $x \in X$ and $\varepsilon > 0$. If $y \in X$ with $d(x, y) < \varepsilon/2$, then we find

$$d((x, x), (y, y)) = d(x, y) + d(x, y) < 2\varepsilon/2 = \varepsilon.$$

Now, let $f, g : X \longrightarrow \mathbb{R}$ be continuous. Then, by what we have seen in part (i) and (ii) and exercise 2, the functions $+ \circ (f \times g) \circ \Delta$ and $\{\cdot\} \circ (f \times g) \circ \Delta$ are compositions of continous functions. Note that

$$+ \circ (f \times g) \circ \Delta(x) = +(f \times g(x, x)) = +(f(x), g(x)) = f(x) + g(x) = (f + g)(x)$$

and similarly $\cdot \circ (f \times g) \circ \Delta = fg$. So we conclude that $f + g$ and fg are compositions of continuous functions and therefore continuous themselves.

Exercise 3.4. Let $X = (X, d)$ and $Y = (Y, d')$ be metric spaces and $f : X \longrightarrow Y$ a map. By a theorem we stated in class on Friday and proved on Monday, f is continuous if and only if the preimage under f of any open (respectively closed) set is open (respectively closed). This suggests an easy way to give many new examples of open and closed sets in a metric space M: write down a function $f : M \longrightarrow \mathbb{R}$, show that it is continuous, and then take the preimage under f of an open or closed set in $\mathbb{R}$ respectively. Using this method:

(i) Fix real positive numbers $a_1, \ldots, a_{k+1} > 0$. Show that the generalized ellipsoid

$$E_{a_1,\ldots,a_{k+1}} = \left\{(x_1, \ldots, x_{k+1}) : \sum_{i=1}^{k+1} a_i x_i^2 = 1\right\}$$

is a closed subset of $\mathbb{R}^{k+1}$.

(ii) Let $V = \{(x_1, x_2, x_3) : x_1^2 + x_2 < 1 \text{ and } (x_3 - 2)^4 > 4\}$. Show that V is an open subset of $\mathbb{R}^3$ with its standard Euclidean metric.

Solution.

(i) Consider the function $f : \mathbb{R}^{k+1} \longrightarrow \mathbb{R}$ with $f(x_1, \ldots, x_{k+1}) = \sum_i a_i x_i^2$. By repeated appliation of exercise 3 part (iii) this function is continuous. Note that $\{1\} \subset \mathbb{R}$ is closed, since any singleton in a metric space is closed. Then $E_{a_1,\ldots,a_{k+1}} = f^{-1}(\{1\})$ is a preimage of a closed set under a continuous functions and therefore closed itself.

(ii) First consider the function $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$ with $f(x_1, x_2, x_3) = x_1^2 + x_2$. Again, by exercise 3 this a continuous function. Since $(-\infty, 1) \subset \mathbb{R}$ is an open set, its preimage

$$f^{-1}((-\infty, 1)) = \{(x_1, x_2, x_3) : x_1^2 + x_2 < 1\}$$

is an open subset of $\mathbb{R}^3$. Similarly, the function $g : \mathbb{R}^3 \longrightarrow \mathbb{R}$ with $g(x_1, x_2, x_3) = (x_3 - 2)^2$ is continuous and $(4, \infty) \subset \mathbb{R}$ is open. Therefore

$$g^{-1}((4, \infty)) = \{(x_1, x_2, x_3) : (x_3 - 2)^2 > 4\}$$

is open. But then $V = f^{-1}((-\infty, 1)) \cap g^{-1}((4, \infty))$ is a finite intersection of open sets and hence open itself.