Math 440 Homework 4

Due Friday, Sept. 22, 2017 by 4 pm

Please remember to write down your name on your assignment.

Please submit your homework to our TA Viktor Kleen, either in his mailbox (in KAP 405) or under the door of his office (KAP 413). You may also e-mail your solutions to Viktor provided:

- you have typed your homework solutions; or
- you are able to produce a very high quality scanned PDF (no photos please!),

1. Let $X := (X, d)$ be a metric space, and $A \subset X$ any subset. A *limit point* of A is a point $x \in X$ such that for every $\epsilon > 0$, $(B_d(x, \epsilon) - \{x\}) \cap A \neq \emptyset$. In words, every ϵ ball around X intersects A in a point other than x (it could also intersect A in x or not, depending on whether $x \in A$).

   (a) Give, with proof, the set of all limit points of the subset $A = \{-1\} \cup (2, 4) \cup \{\frac{1}{n} | n \in \mathbb{N}\} \subset \mathbb{R}$.

   (b) Let $y \in X$ be a limit point of a set A. Show that there exists a sequence of points $\{x_n\}_{n \in \mathbb{N}}$, with each $x_n \in A$, converging to $y \in X$. In particular, there is an inclusion

   $$\{\text{limit points of } A \text{ in } X\} \subset \{\text{limits in } X \text{ of convergent sequences in } A\}.$$

   Is this inclusion always an equality? That is, for any X and A, is every limit in X of a convergent sequence in A also a limit point in the sense described above? (Justify: that is, either prove there is an equality or find an example of an element on the RHS which is not on the LHS, for some X and A)

   (c) Prove that, for any A, X, the set $\bar{A} := A \cup \{\text{limit points of } A\}$ is always a *closed* subset of X (that is, that $X - A - \{\text{limit points of } A\}$ is open). The set $\bar{A}$ is often called the *closure* of A in X. (*Note*: We will give a different definition of closure in class, and prove it is equivalent to this one).

2. Let $X = \{a, b, c, d\}$, and let $\mathcal{T} = \{\emptyset, X, \{a, b, d\}, \{b, c, d\}, \{b, d\}, \{d\}\}$.

   (a) Prove that $X := (X, \mathcal{T})$ is a topological space; that is that $\mathcal{T}$ is a topology.

   (b) Prove that X is not metrizable. That is, prove there does not exist a metric $d : X \times X \to [0, \infty)$ such that $\mathcal{T} = \mathcal{T}_d$. (**Hint**: Here is one approach: find a subset Y of X which would have to be open with respect to any metric on X, that does not lie in $\mathcal{T}$).

3. Let $X = (X, \mathcal{T})$ be a topological space, and $A \subset X$ any subset. Suppose that for every $x \in A$, there exists an open set $U \subset X$ containing x which is contained in A. Then, prove that A is open in X.

4. Let $X = \mathbb{N} \cup \{\star\}$[1] , and consider the following collection of subsets of X:

$$\mathcal{T} = \{A \mid A \subset \mathbb{N} \text{ arbitrary}\} \cup \{B \cup \{\star\} \mid B \subset \mathbb{N} \text{ and } \mathbb{N} - B \text{ is a finite set}\}.$$

To rephrase, we say a subset of U is contained in $\mathcal{T}$ iff one of the following conditions holds:

- Either $\star \notin U$ (and U is otherwise arbitrary, e.g., any subset of $\mathbb{N} \subset X$); OR
- $\star \in U$ AND the complement $X - U$ is a *finite* set.

Note: Examples of sets in $\mathcal{T}$ include: $\mathbb{N}$, $\{\text{even numbers}\}$, $\{1\}$, $\{2, 3, 4, 5, \ldots, \} \cup \{\star\}$, $X - \{1, 3, 5, 10\}$. Some examples of sets *not* in $\mathcal{T}$ include: $\{\star\}$, $\{2, 4, \star\}$, $\{\text{all even numbers}\} \cup \{\star\}$.

(a) Prove that $\mathcal{T}$ is a topology on X, and hence that $X := (X, \mathcal{T})$ is a topological space. (**Note**: When showing that $\mathcal{T}$ is closed under arbitrary unions and finite intersections, you may use the following facts without justification: any subset of a finite set is always finite, and any finite union of finite sets remains finite).

(b) Note that there is a natural inclusion $i : \mathbb{N} \to X$. Prove i is continuous, as a map from $\mathbb{N}$ with its discrete topology $\mathcal{T}_{discrete}$ to X with the topology $\mathcal{T}$ defined above.

(c) Let (Y, d) be a metric space, which in particular induces a topological space $Y : (Y, \mathcal{T}_d)$. Show that, for each point $y \in Y$, there is a bijection between the following two sets:

$$A = \{\text{Continuous maps } f : (X, \mathcal{T}) \to (Y, \mathcal{T}_d) \text{ with } f(\star) = y\}$$

and

$$B = \{\text{Convergent sequences in } Y, \{x_n\}_{n \in \mathbb{N}}, \text{ with limit } y\}.$$

Note: Note that we are using the definition of continuous maps valid for abstract topological spaces: that is, f is continuous means that the preimage of every open set is open. In particular, since $(X, \mathcal{T})$ is not a metric space, you cannot use facts proved for metric spaces (such as "continuity implies sequential continuity")

Hint: Let $f \in A$ be a continuous map with $f(\star) = y$. Then, by by composing with $i : \mathbb{N} \to X$, we get a map $f \circ i : \mathbb{N} \to Y$, which is the same as a *sequence* in Y $\{x_n\}_{n \in \mathbb{N}}$. Why is this sequence necessarily convergent with limit y? (you'll want to use continuity of f). Once you establish this, it will follow that $f \mapsto f \circ i$ indeed defines a map $A \to B$. Next you'll want to argue its a bijection.

Hint 2: It may be helpful to first prove, and then use, the following Proposition, characterizing convergent sequences:

PROPOSITION. *Let (Y, d) be any metric space, and $\{x_n\}_{n \in \mathbb{N}}$ any sequence of points in Y. Then, the following three statements are equivalent:*

(i) The sequence $\{x_n\}$ converges to the point $y \in Y$.

[1]Note: In an earlier version of this problem, we used the symbol ∞ instead of $\star$. We switched to $\star$ to make clear that we are just adding an extra abstract element to $\mathbb{N}$. You are welcome to use either "∞" or "$\star$" in your write-up, depending on what you prefer (and what you may have already begun using).

(ii) For any $\epsilon > 0$, the ball $B_d(y, \epsilon)$ contains all but finitely many of the elements x_n. (meaning: the set $\{n \in \mathbb{N} | x_n \notin B_d(y, \epsilon)\}$ is finite)

(iii) For any open set U containing y, U contains all but finitely many of the elements x_n. (meaning: the set $\{n \in \mathbb{N} | x_n \notin U\}$ is finite)

(of course, we will award partial credit if you use this Proposition without proof to solve (c)).

Remark: In light of the above property of X (that continuous maps out of X are the same as convergent sequences with a specified limit), we may sometimes call the topological space X the *abstract convergent sequence.*