Homework 4

Exercise 4.1. Let $X = (X, d)$ be a metric space, and $A \subset X$ any subset. A *limit point* of A is a point $x \in X$ such that every $\varepsilon > 0$, we have $(B_d(x, \varepsilon) \setminus \{x\}) \cap A \neq \emptyset$. In words, every ε−ball around x intersects A in a point other than x.

(i) Give, with proof, the set of all limit points of the subset $A = \{-1\} \cup (2, 4) \cup \{1/n : n \in \mathbb{N}\} \subset \mathbb{R}$.

(ii) Let $y \in X$ be a limit point of a set A. Show that there exists a sequence of points $\{x_n\}_{n\in\mathbb{N}}$, with each $x_n \in A$, converging to $y \in X$. In particular, there is an inclusion

$$\{\text{limit points of } A \text{ in } X\} \subset \{\text{limits in } X \text{ of convergent sequences in } A\}.$$

Is this inclusion always an equality? That is, for any X and A, is every limit in X of a convergent sequence in A also a limit point in the sense described above?

(iii) Prove that, for any A, X, the set $\overline{A} = A \cup \{\text{limit points of } A\}$ is always a *closed* subset of X. The set $\overline{A}$ is often called the *closure* of A in X.

Solution.

(i) The set of limit points is $[2, 4] \cup \{0\}$: First, we check that every $x \in [2, 4] \cup \{0\}$ is indeed a limit point of A. There are several cases:

- (i) $x \in (2, 4)$: Let $\varepsilon > 0$ and write $r = \min\{\varepsilon, 1 - |x - 3|\}$. Then $(x - r, x + r) \subset B(x, \varepsilon) \cap (2, 4)$. Consequently, for example $x + r/2 \in (B(x, \varepsilon) \setminus \{x\}) \cap A$.
- (ii) $x = 2$: If $\varepsilon > 0$ and $r = \min\{\varepsilon, 2\}$, then $2 + r/2 \in B(2, \varepsilon) \setminus \{2\}$ and $2 + r/2 \in A$.
- (iii) $x = 4$: Similarly, if $\varepsilon > 0$ and $r = \min\{\varepsilon, 2\}$, then $4 - r/2 \in B(4, \varepsilon) \setminus \{4\}$ and $4 - r/2 \in A$.
- (iv) $x = 0$: Suppose $\varepsilon > 0$. Let $N \in \mathbb{N}$ be an integer large enough that $N > 1/\varepsilon$. Then we have $1/N \in A$ and $0 < 1/N < \varepsilon$, i. e. $1/N \in (B(0, \varepsilon) \setminus \{0\}) \cap A$.

Next, we need to verify that every limit point of A is in $[2, 4] \cup \{0\}$. For this suppose $x \in \mathbb{R} \setminus ([2, 4] \cup \{0\})$. We again distinguish several cases:

- (i) $x < -1$: If $\varepsilon < |x + 1|$, observe that $B(x, \varepsilon) \cap A = \emptyset$.
- (ii) $x = -1$: We have $(B(-1, 1/2) \setminus \{-1\}) \cap A = \emptyset$.
- (iii) $-1 < x < 0$: If $\varepsilon < \min\{|x + 1|, |x|\}$, then $B(x, \varepsilon) \cap A = \emptyset$.
- (iv) $0 < x < 1$ and $x = \frac{1}{n} \in A$: If $\varepsilon < \frac{1}{n+1}$, then $B(x, \varepsilon) \cap A = \{x\}$ because $(\frac{1}{n+1}, x) \cup (x, \frac{1}{n-1}) \not\subset A$.
- (v) $0 < x < 1$ and $x \notin A$: In this case, there is some $n \in \mathbb{N}$ such that $\frac{1}{n+1} < x < \frac{1}{n}$; in fact $n = \lfloor 1/x \rfloor$. Then, for any $\varepsilon < \min\{|x - \frac{1}{n+1}|, |x - \frac{1}{n}|\}$, we have $B(x, \varepsilon) \cap A = \emptyset$.
- (vi) $x = 1$: We have $B(1, 1/3) \cap A = \{1\}$.
- (vii) $1 < x < 2$: If $\varepsilon < \min\{|x - 1|, |x - 2|\}$, then $B(x, \varepsilon) \cap A = \emptyset$.
- (viii) $4 < x$: For every $\varepsilon < |x - 4|$ we have $B(x, \varepsilon) \cap A = \emptyset$.

So, no $x \notin [2, 4] \cup \{0\}$ can be a limit point of A.

(ii) Suppose $y \in X$ is a limit point of A. Then for all $n \in \mathbb{N}$ the set $B_d(y, \frac{1}{n}) \cap A$ is nonempty; so pick some $x_n \in B_d(y, \frac{1}{n}) \cap A$. Then $\{x_n\}_n$ converges to y: Indeed, suppose $\varepsilon > 0$ and $N \in \mathbb{N}$ with $N > \frac{1}{\varepsilon}$. Then $d(y, x_n) < \frac{1}{n} \leq \frac{1}{N} < \varepsilon$ whenever $n \geq N$.

This shows that any limit point of A is a limit in X of a convergent sequence in A. The converse is false. For example, -1 is not a limit point of $\{-1\} \subset \mathbb{R}$, but the constant sequence $\{-1\}_n$ converges to -1 in $\mathbb{R}$.

(iii) Suppose $x \notin \overline{A}$. Then there must be some $\varepsilon > 0$ such that $(B_d(x, \varepsilon) \cap A) \setminus \{x\} = \emptyset$. Since in particular $x \notin A$ it must be the case that $B_d(x, \varepsilon) \cap A = \emptyset$. Now, suppose for contraction that some limit point $y \in X$ of A were to lie in $B_d(x, \varepsilon)$. Let $\delta > 0$ be small enough such that $B_d(y, \delta) \subset B_d(x, \varepsilon)$; such a number δ exists because $B_d(x, \varepsilon)$ is an open subset of X. But then $B_d(y, \delta) \cap A \subset B_d(x, \varepsilon) \cap A = \emptyset$ contradicting y being a limit point.

Hence, we conclude that $B_d(x, \varepsilon) \cap \overline{A} = \emptyset$ or in other words $B_d(x, \varepsilon) \subset X \setminus \overline{A}$. Since $x \in X \setminus \overline{A}$ was chosen arbitrarily this implies that $X \setminus \overline{A}$ is open, that is, $\overline{A}$ is closed.

Exercise 4.2. Let $X = \{a, b, c, d\}$, and let $\mathcal{T} = \{\emptyset, X, \{a, b, d\}, \{b, c, d\}, \{b, d\}, \{d\}\}$.

(i) Prove that $X = (X, \mathcal{T})$ is a topological space; that is, that $\mathcal{T}$ is a topology.
(ii) Prove that X is not metrizable. That is, prove there does not exist a metric $d \colon X \times X \longrightarrow [0, \infty)$ such that $\mathcal{T} = \mathcal{T}_d$.

Solution.

(i) From the definition of $\mathcal{T}$ we immediately have $\emptyset, X \in \mathcal{T}$.
Suppose we have a potentially infinite collection of sets $U_i \in \mathcal{T}$, $i \in I$. Since there are only finitely many elements of $\mathcal{T}$, the union $\bigcup_{i\in I} U_i$ can actually be written as a union of finitely many elements of $\mathcal{T}$. This means we only have to prove that the union of any two elements of $\mathcal{T}$ is again in $\mathcal{T}$. First, $A \cup \emptyset = A$ and $A \cup X = X$ for any subset $A \subset X$. Furthermore,

$$\begin{aligned} \{a, b, d\} \cup \{b, c, d\} &= X \in \mathcal{T} \\ \{a, b, d\} \cup \{b, d\} &= \{a, b, d\} \in \mathcal{T} \\ \{a, b, d\} \cup \{d\} &= \{a, b, d\} \in \mathcal{T} \\ \{b, c, d\} \cup \{b, d\} &= \{b, c, d\} \in \mathcal{T} \\ \{b, c, d\} \cup \{d\} &= \{b, c, d\} \in \mathcal{T} \end{aligned}$$

and

$$\{b, d\} \cup \{d\} = \{b, d\} \in \mathcal{T}.$$

For intersections of finitely many sets in $\mathcal{T}$, we compute

$$\begin{aligned} \{a, b, d\} \cap \{b, c, d\} &= \{b, d\} \in \mathcal{T} \\ \{a, b, d\} \cap \{b, d\} &= \{b, d\} \in \mathcal{T} \\ \{a, b, d\} \cap \{d\} &= \{d\} \in \mathcal{T} \\ \{b, c, d\} \cap \{b, d\} &= \{b, d\} \in \mathcal{T} \\ \{b, c, d\} \cap \{d\} &= \{d\} \in \mathcal{T} \end{aligned}$$

and

$$\{b, d\} \cap \{d\} = \{d\} \in \mathcal{T}.$$

Also, for any subset $A \subset X$ we have $\emptyset \cap A = \emptyset$ and $X \cap A = A$. All of this put together shows that $\mathcal{T}$ is a topology on X.

(ii) Suppose d is a metric on X. Then the set $\{d\} \subset X$ would be closed because singletons are closed in any metric space. Consequently, the complement $X \setminus \{d\} = \{a, b, c\}$ would have to be open in the topology induced by $\{d\}$. But $\{a, b, c\} \notin \mathcal{T}$. Hence, the topology induced by d cannot be $\mathcal{T}$ and therefore $\mathcal{T}$ can't be metrizable.

Exercise 4.3. Let $X = (X, \mathcal{T})$ be a topological space, and $A \subset X$ any subset. Suppose that for every $x \in A$, there exists an open set $U \subset X$ containing x which is contained in A. Then, prove that A is open in X.

Solution. For any $x \in A$ pick an open set $U_x \subset A$ with $x \in U_x$. Then $\bigcup_{x\in A} U_x = A$: Since any U_x is a subset of A, there union $\bigcup_{x\in A} U_x$ is still a subset of A. Conversely, if $y \in A$, then $y \in U_y \subset \bigcup_{x\in A} U_x$. But then $A = \bigcup_{x\in A} U_x$ is a union of open subsets of X and therefore open itself.

Proposition. *Let (Y, d) be any metric space, and $\{x_n\}_{n\in\mathbb{N}}$ any sequence of points in Y. Then the following statements are equivalent:*

(i) *The sequence $\{x_n\}$ converges to $y \in Y$.*
(ii) *For any $\varepsilon > 0$, the ball $B_d(y, \varepsilon)$ contains all but finitely many of the x_n.*
(iii) *Any open set U containing y also contains all but finitely many of the x_n.*

Proof. First, suppose that $\{x_n\}$ converges to $y \in Y$ and let $\varepsilon > 0$. Then there is some natural number $N \in \mathbb{N}$ such that $d(x_n, y) < \varepsilon$ for all $n > N$. This means that we have an inclusion $\{x_n : n > N\} \subset B_d(y, \varepsilon)$ of sets and therefore $\{n \in \mathbb{N} : x_n \notin B_d(y, \varepsilon)\} \subset \{n \in \mathbb{N} : n \leq N\}$. But the latter set is finite. So the set $\{n \in \mathbb{N} : x_n \notin B_d(y, \varepsilon)\}$ must also be finite; in other words, $B_d(y, \varepsilon)$ contains all but finitely many of the x_n.

Next, suppose (ii) holds and let U be an open set containing y. Because U is an open subset of the metric space Y, there is some $\varepsilon > 0$ such that $B_d(y, \varepsilon) \subset U$. Then, by (ii), $B_d(y, \varepsilon)$ contains all but finitely many of th x_n. But because $B_d(y, \varepsilon) \subset U$ this implies that U contains all but finitely many of the x_n as well.

Finally, suppose (iii) holds and $\varepsilon > 0$. Then $B_d(y, \varepsilon)$ is an open set ontaining y and therefore the set $\{n \in \mathbb{N} : x_n \notin B_d(y, \varepsilon)\}$ is finite. Consequently, it has a maximum element N. So, if $n > N$ then we must have $x_n \in B_d(y, \varepsilon)$. Since $\varepsilon > 0$ was arbitary this means that $\{x_n\}$ converges to $y \in Y$. □

Exercise 4.4. Let $X = \mathbb{N} \cup \{\infty\}$, and consider the following collection of subsets of X:

$$\mathcal{T} = \{A : A \subset \mathbb{N}\} \cup \{B \cup \{\infty\} : B \subset \mathbb{N} \text{ is the complement of a finite set}\}.$$

(i) Prove that $\mathcal{T}$ is a topology on X, and hence that $X = (X, \mathcal{T})$ is a topological space.
(ii) Note that there is a natural inclusion $i : \mathbb{N} \longrightarrow X$. Prove that i is continuous, as a map from $\mathbb{N}$ with its discrete topology and X with the topology above.
(iii) Let (Y, d) be a metric space, which in particular induces a topological space $(Y, \mathcal{T}_d)$. Show that, for each point $y \in Y$, there is a bijection between the following two sets:

$$A = \{\text{continuous maps } f : X \longrightarrow Y \text{ with } f(\infty) = y\}$$

and

$$B = \{\text{convergent sequences } \{x_n\}_{n\in\mathbb{N}} \text{ with limit } y\}.$$

Solution.

(i) First, $\emptyset \subset \mathbb{N}$ and therefore $\emptyset \in \mathcal{T}$. Also, $X = \mathbb{N} \cup \{\infty\}$ and $\mathbb{N} \setminus \mathbb{N} = \emptyset$ is finite, so $X \in \mathcal{T}$ as well. Now, suppose $\{U_i\}_{i\in I}$ is a family of sets $U_i \in \mathcal{T}$. If $\infty \notin \bigcup_{i\in I} U_i$, then $\bigcup_{i\in I} U_i \subset \mathbb{N}$ and so $\bigcup_{i\in I} U_i \in \mathcal{T}$. Hence, suppose $\infty \in \bigcup_{i\in I} U_i$. Then there is some $j \in I$ with $\infty \in U_j$ and because $U_j \in \mathcal{T}$ we conclude that $X \setminus U_j$ is finite. But then $X \setminus \bigcup_{i\in I} U_i = \bigcap_{i\in I} X \setminus U_i \subset X \setminus U_j$ is also finite and we can conclude that $\bigcup_{i\in I} U_i \in \mathcal{T}$.
Finally, let $U_1, \ldots, U_n$ be a finite collection of sets $U_i \in \mathcal{T}$. Again, if $\infty \notin \bigcap_{i=1}^n U_i$, then $\bigcap_{i=1}^n U_i \subset \mathbb{N}$ and $\bigcap_{i=1}^n U_i \in \mathcal{T}$. So, suppose that $\infty \in \bigcap_{i=1}^n U_i$. Then $\infty \in U_i$ for all $i = 1, \ldots, n$ and because $U_i \in \mathcal{T}$ the complements $X \setminus U_i$ are finite for all i. But then $X \setminus \bigcap_{i=1}^n U_i = \bigcup_{i=1}^n X \setminus U_i$ is union of finitely many finite sets and therefore finite itself. Hence, $\bigcap_{i=1}^n U_i \in \mathcal{T}$.
(ii) For *any* subset $U \subset X$ the preimage $i^{-1}(U) \subset \mathbb{N}$ is open in $\mathbb{N}$ because for the discrete topology any subset is open. In particular, $i^{-1}(U) \subset \mathbb{N}$ is open for $U \subset X$ an open subset.
(iii) First, we define a map $\varphi : A \longrightarrow B$. Take any $f \in A$, that is, f is a continuous map $X \longrightarrow Y$ with $f(\infty) = y$. Consider the sequence $\{f(n)\}_{n\in\mathbb{N}}$. We claim that this sequence converges to y. For this, let $U \subset Y$ be an open set containing y. Because f is continuous the preimage $f^{-1}(U)$ is open and because $f(\infty) = y \in U$ we also have $\infty \in f^{-1}(U)$. Hence, by the definition of the topology on X there are only finitely many $n \in \mathbb{N}$ with $n \notin f^{-1}(U)$, that is $f(n) \notin U$. So U contains all but finitely many of the $x_n = f(n)$. Because U was arbitary, by the proposition we can conclude that $\{x_n\}$ converges to y and therefore $\{x_n\} \in B$. So, by setting $\varphi(f) = \{f(n)\}_{n\in\mathbb{N}}$ we obtain a well-defined map $\varphi : A \longrightarrow B$.
Conversely, we would like to define a map $\psi : B \longrightarrow A$ with $\psi(\{x_n\}_{n\in\mathbb{N}})$ being the function $n \longmapsto x_n$, $\infty \longmapsto y$. For this to make sense, we need to check that the function $f : X \longrightarrow Y$ with $f(n) = x_n$ and $f(\infty) = y$ is in fact an element of A, that is, we need to check that this function is continuous. For this, suppose $U \subset Y$ is open. If $y \notin U$ then $\infty \notin f^{-1}(U)$ and $f^{-1}(U)$ is open in X. So, assume that $y \in U$. Because $\{x_n\}$ converges to y, by the proposition, U contains all but finitely many of the x_n. That is, the

set $\{n \in \mathbb{N} : x_n \notin U\}$ is finite; but $\{n \in \mathbb{N} : x_n \notin U\} = X \smallsetminus f^{-1}(U)$ and this set being finite means that $f^{-1}(U)$ is open in X. So $f : X \longrightarrow Y$ indeed is a continuous map.
In summary, we have two maps $\varphi\colon A \longrightarrow B$ and $\psi\colon B \longrightarrow A$. To see that they are inverses of each other, we compute

$$\varphi(\psi(\{x_n\})) = \{\psi(\{x_n\})(n)\}_{n\in\mathbb{N}} = \{x_n\}_{n\in\mathbb{N}}$$
$$\psi(\varphi(f))(n) = \psi(\{f(n)\})(n) = f(n)$$

for $f \in A$ and $\{x_n\} \in B$. So, $\varphi \circ \psi = \mathrm{id}_B$ and $\psi \circ \varphi = \mathrm{id}_A$ and we can conclude in particular that $\varphi\colon A \longrightarrow B$ is a bijection.