Math 440 Homework 5

Due Monday, Oct. 2, 2017 by 4 pm

Please remember to write down your name on your assignment.

Please submit your homework to our TA Viktor Kleen, either in his mailbox (in KAP 405) or under the door of his office (KAP 413). You may also e-mail your solutions to Viktor provided:

- you have typed your homework solutions; or
- you are able to produce a very high quality scanned PDF (no photos please!),

1. Let A, B and A_i, (for $i \in I$ some index set) denote subsets of a topological space $X := (X, \mathcal{T})$. Prove the following:
   (a) If $A \subset B$, then $\mathrm{cl}(A) \subset \mathrm{cl}(B)$.

   (b) $\mathrm{cl}(A \cup B) = \mathrm{cl}(A) \cup \mathrm{cl}(B)$.

2. (a) Show that in a metric space (X, d), the closure of an open ball $B(x, r)$ is contained in the closed ball $\bar{B}(x, r)$.

   (b) Give an example (with proof) where $\bar{B}(x, r)$ is different from the closure $\mathrm{cl}(B(x, r))$.

3. **The boundary of a set.** If A is a subset of a topological space X, define the *boundary of* A to be the set

$$\partial A := \mathrm{cl}(A) - \mathrm{int}(A)$$

   That is, the boundary of A is the difference between the closure of A and the interior of A. Prove that
   (a) ∂A is closed for any set $A \subset X$. *Hint*: it may be helpful to prove the following formula first:

$$\partial A = \mathrm{cl}(A) \cap \mathrm{cl}(X - A).$$

   This in turn might benefit from the fact, proven in class, that $\mathrm{cl}(X - A) = X - \mathrm{int}(A)$.

   (b) $A \cup \partial A = \mathrm{cl}(A)$, for any A.

   (c) $A - \partial A = \mathrm{int}(A)$, for any X. (note: ∂A is not necessarily strictly contained in A; as usual the notation $A - \partial A$ refers to the set of points in A which are not in ∂A).

4. Let $A = (\mathbb{Q} \cap (0, 1)) \cup \{2\} \cup (3, 5]$, thought of as a subset of $\mathbb{R}$ with its standard topology. Compute with proof the sets $\mathrm{cl}(A)$, $\mathrm{int}(A)$ and ∂A.

   (*Remark*: If it is helpful, you may use without proof the following fact: any open interval (a, b) in $\mathbb{R}$ (with $a < b$) contains infinitely many rational and irrational numbers.)

5. Consider $Y = \mathbb{Q}$, endowed with the subspace topology[1] for the inclusion $\mathbb{Q} \subset \mathbb{R}$ (where $\mathbb{R}$ carries its standard topology). Let $A = \{p \in \mathbb{Q} | 2 < p^2 < 3\} \subset \mathbb{Q} \subset \mathbb{R}$.

   First, we note that A is an open subset of $\mathbb{Q}$. Indeed, $A = U \cap \mathbb{Q}$, where $U = \{p \in \mathbb{R} | 2 < p^2 < 3\}$ is an open subset of $\mathbb{R}$ (why?). Therefore, by definition, since A is the intersection of an open subset in $\mathbb{R}$ with $\mathbb{Q}$, A is open in the subspace topology of $\mathbb{Q}$.

   (a) Prove, on the other hand, that A is not open in $\mathbb{R}$. (*Hint*: it may be helpful to again use the remark from # 4).

   (b) What is the closure of A in $\mathbb{R}$ (denoted $\mathrm{cl}_{\mathbb{R}}(A)$)? (justify with proof — again the remark from #4 may be helpful)

   (c) What is the closure of A in $\mathbb{Q}$ (denoted $\mathrm{cl}_{\mathbb{Q}}(A)$)? (justify with proof)

   Hint: you may use, without proof, the fact that $\sqrt{2}$ and $\sqrt{3}$ are irrational numbers.

[1] Let $X := (X, \mathcal{T}_X)$ be a topological space and $C \subset X$ any subset. In class on Wednesday, we define a topology on C, called the *subspace topology*, by the following prescription: $\mathcal{T}_C := \{U \subset C |$ there exists an open set $V \subset X$ such that $V \cap C = U\}$