Homework 5

Exercise 5.1. Let A and B denote subsets of a topological space $X = (X, \mathcal{T})$. Prove the following:

(i) If $A \subset B$, then $\mathrm{cl}(A) \subset \mathrm{cl}(B)$.
(ii) $\mathrm{cl}(A \cup B) = \mathrm{cl}(A) \cup \mathrm{cl}(B)$.

Solution.

(i) Note that $\mathrm{cl}(B)$ is a closed set and $A \subset B \subset \mathrm{cl}(B)$. But $\mathrm{cl}(A)$ is the intersection of all closed sets containing A and $\mathrm{cl}(B)$ being one of them we conclude $\mathrm{cl}(A) \subset \mathrm{cl}(B)$.

(ii) Since finite unions of closed sets remain closed, the set $\mathrm{cl}(A) \cup \mathrm{cl}(B)$ is closed in X. Furthermore $A \subset \mathrm{cl}(A) \subset \mathrm{cl}(A) \cup \mathrm{cl}(B)$ and $B \subset \mathrm{cl}(B) \subset \mathrm{cl}(A) \cup \mathrm{cl}(B)$, so $A \cup B \subset \mathrm{cl}(A) \cup \mathrm{cl}(B)$. Again, $\mathrm{cl}(A \cup B)$ is the intersection of all closed sets containing $A \cup B$ and $\mathrm{cl}(A) \cup \mathrm{cl}(B)$ is such a set, so we conclude $\mathrm{cl}(A \cup B) \subset \mathrm{cl}(A) \cup \mathrm{cl}(B)$.
On the other hand, from part (i) and $A \subset A \cup B$ we can conclude $\mathrm{cl}(A) \subset \mathrm{cl}(A \cup B)$ and similarly $\mathrm{cl}(B) \subset \mathrm{cl}(A \cup B)$. Therefore, $\mathrm{cl}(A) \cup \mathrm{cl}(B) \subset \mathrm{cl}(A \cup B)$ and combining this with the previous paragraph we conclude that $\mathrm{cl}(A \cup B) = \mathrm{cl}(A) \cup \mathrm{cl}(B)$.

Exercise 5.2.

(i) Show that in a metric space (X, d), the closure of an open ball $B(x, r)$ is contained in the closed ball $\overline{B}(x, r)$.
(ii) Give an example (with proof) where $\overline{B}(x, r)$ is different from the closure $\mathrm{cl}(B(x, r))$.

Solution.

(i) First, observe that the closed ball $\overline{B}(x, r)$ is a closed set: Consider the function $f \colon X \longrightarrow \mathbb{R}$ with $f(y) = d(x, y)$. This function is continuous[1] and $\overline{B}(x, r) = f^{-1}([0, r])$ is the preimage of a closed set in $\mathbb{R}$.
Now, $\overline{B}(x, r)$ is a closed set containing $B(x, r)$ and $\mathrm{cl}(B(x, r))$ is the intersection of all such sets. So $\mathrm{cl}(B(x, r)) \subset \overline{B}(x, r)$.

(ii) Consider any set X containing at least 2 elements and equip it with the discrete metric. Let $x \in X$. Then $B(x, 1) = \{x\}$ is closed in X and therefore $\mathrm{cl}(B(x, 1)) = B(x, 1) = \{x\}$. But $\overline{B}(x, 1) = X \neq \{x\}$ because we assumed X contains at least 2 elements.

Exercise 5.3. If A is a subset of a topological space X, define the *boundary of* A to be the set

$$\partial A = \mathrm{cl}(A) \setminus \mathrm{int}(A).$$

That is, the boundary of A is the difference between the closure of A and the interior of A. Prove that

(i) The boundary ∂A is closed for any set $A \subset X$.
(ii) $A \cup \partial A = \mathrm{cl}(A)$ for any A.
(iii) $A \setminus \partial A = \mathrm{int}(A)$ for any A.

Solution.

(i) Note that $\partial A = \mathrm{cl}(A) \cap (X \setminus \mathrm{int}(A))$ and $\mathrm{cl}(A)$ is closed in X. Also, $\mathrm{int}(A)$, being a union of open sets, is open and therefore $X \setminus \mathrm{int}(A)$ is closed in X. Hence, ∂A is an intersection of two closed sets, and therefore closed itself.

(ii) First, by definition $A \subset \mathrm{cl}(A)$ and $\partial A \subset \mathrm{cl}(A)$ and therefore $A \cup \partial A \subset \mathrm{cl}(A)$. On the other hand, because $\mathrm{int}(A) \subset A$, there is an inclusion $\mathrm{cl}(A) \setminus A \subset \mathrm{cl}(A) \setminus \mathrm{int}(A)$ and because $A \subset \mathrm{cl}(A)$ we have $\mathrm{cl}(A) = (\mathrm{cl}(A) \setminus A) \cup A \subset (\mathrm{cl}(A) \setminus \mathrm{int}(A)) \cup A = A \cup \partial A$.

(iii) Writing $C^c = X \setminus C$, we have $A \setminus \partial A = A \cap (\mathrm{cl}(A) \cap \mathrm{int}(A)^c)^c = (A \cap \mathrm{cl}(A)^c) \cup (A \cap \mathrm{int}(A))$. But $A \subset \mathrm{cl}(A)$ and $\mathrm{int}(A) \subset A$, so $A \cap \mathrm{cl}(A)^c = \emptyset$ and $A \cap \mathrm{int}(A) = \mathrm{int}(A)$. Therefore, $A \setminus \partial A = \emptyset \cup \mathrm{int}(A) = \mathrm{int}(A)$.

[1] For every $\varepsilon > 0$ if $d(y, y') < \varepsilon$, then $f(y') - f(y) = d(x, y') - d(x, y) \leq d(y, y') < \varepsilon$ and $f(y) - f(y') = d(x, y) - d(x, y') \leq d(y, y') < \varepsilon$; i. e. $|f(y') - f(y)| < \varepsilon$.

Exercise 5.4. Let $A = (\mathbb{Q} \cap (0,1)) \cup \{2\} \cup (3,5]$, thought of as a subset of $\mathbb{R}$ with its standard topology. Compute with proof the sets $\mathrm{cl}(A)$, $\mathrm{int}(A)$ and ∂A.
Solution. From problem 1 we know that

$$\mathrm{cl}(A) = \mathrm{cl}((\mathbb{Q} \cap (0,1)) \cup \{2\} \cup (3,5]) = \mathrm{cl}(\mathbb{Q} \cap (0,1)) \cup \mathrm{cl}(\{2\}) \cup \mathrm{cl}((3,5]).$$

Because $\{2\}$ is closed already, we immediately see $\mathrm{cl}(\{2\}) = \{2\}$. We will argue that $\mathrm{cl}(\mathbb{Q} \cap (0,1)) = [0,1]$ and $\mathrm{cl}((3,5]) = [3,5]$ so that $\mathrm{cl}(A) = [0,1] \cup \{2\} \cup [3,5]$. First, $[0,1]$ and $[3,5]$ are closed sets containing $\mathbb{Q} \cap (0,1)$ and $(3,5]$ respectively, so $\mathrm{cl}(\mathbb{Q} \cap (0,1)) \subset [0,1]$ and $\mathrm{cl}((3,5]) \subset [3,5]$.

Conversely, suppose first that $x \in (0,1)$ and let $U \subset \mathbb{R}$ be an open set containing x. Then $(0,1) \cap U$ is open, so there is an $\varepsilon > 0$ such that $(x-\varepsilon, x+\varepsilon) \subset (0,1) \cap U$. But every open interval in $\mathbb{R}$ contains infinitely many rational numbers. Hence, $(x-\varepsilon, x+\varepsilon) \cap \mathbb{Q} \neq \emptyset$ and we conclude that in fact $U \cap \mathbb{Q} \cap (0,1) \neq \emptyset$. This implies that $x \in \mathrm{cl}(\mathbb{Q} \cap (0,1))$. Now, $\{\frac{1}{n+2}\}_{n\in\mathbb{N}}$ is a sequence in $\mathbb{Q} \cap (0,1)$ and $\lim_n \frac{1}{n+2} = 0$ in $\mathbb{R}$. Hence we must have $0 \in \mathrm{cl}(\mathbb{Q} \cap (0,1))$. Similarly, $\lim_n 1 - \frac{1}{n+2} = 1$ and therefore $1 \in \mathrm{cl}(\mathbb{Q} \cap (0,1))$. In conclusion, $\mathrm{cl}(\mathbb{Q} \cap (0,1)) = [0,1]$ as claimed.

As for $(3,5]$, we already know $(3,5] \subset \mathrm{cl}((3,5])$, so we only need to prove $3 \in \mathrm{cl}((3,5])$. For this, just observe that $\lim_n 3 + \frac{1}{n} = 3$ and $3 + \frac{1}{n} \in (3,5]$ for all $n \in \mathbb{N}$.

To compute the interior of A, let U be any open set with $U \subset A$. Suppose for contradition that some $x \in \mathbb{Q} \cap (0,1)$ were contained in U. Then there would be some $\varepsilon > 0$ with $(x-\varepsilon, x+\varepsilon) \subset U$. But then $(x-\varepsilon, x+\varepsilon) \cap (0,1) \subset U$ as well and the latter would have to contain infinitely many irrational numbers. If $y \in (x-\varepsilon, x+\varepsilon) \cap (0,1)$ is irrational, then $y \notin A$ contradicting our assumption that $U \subset A$. We conclude that $U \cap \mathbb{Q} \cap (0,1) = \emptyset$. Furthermore, we also have $2 \notin U$ since otherwise there would again be some $\varepsilon > 0$ with $(2-\varepsilon, 2+\varepsilon) \subset U$. But $(2-\varepsilon, 2+\varepsilon)$ contains infinitely many points outside of A, for example $2 + \frac{1}{n} \notin A$ for $n > 1/\varepsilon$. Similarly, $5 \in U$ is impossible because otherwise there would again be some $\varepsilon > 0$ with $(5-\varepsilon, 5+\varepsilon) \subset A$. But $5 + \frac{1}{n} \notin A$ for $n > 1/\varepsilon$.

In summary, any open set U contained in A satisfies the stronger inclusion $U \subset (3,5)$. But $(3,5)$ is an open set contained in A, so we conclude that in fact $(3,5) = \mathrm{int}(A)$.

Now, the boundary of A is easily computed as

$$\partial A = \mathrm{cl}(A) \setminus \mathrm{int}(A) = ([0,1] \cup \{2\} \cup [3,5]) \setminus (3,5) = [0,1] \cup \{2,3,5\}.$$

Exercise 5.5. Consider $Y = \mathbb{Q}$, endowed with the subspace topology for the inclusion $\mathbb{Q} \subset \mathbb{R}$ (where $\mathbb{R}$ carries its standard topology). Let $A = \{p \in \mathbb{Q} : 2 < p^2 < 3\} \subset \mathbb{Q} \subset \mathbb{R}$.

First, we note that A is an open subset of $\mathbb{Q}$. Indeed, $A = U \cap \mathbb{Q}$, where $U = \{p \in \mathbb{R} : 2 < p^2 < 3\}$ is an open subset of $\mathbb{R}$. Therefore, by definition, since A is the intersection of an open subset in $\mathbb{R}$ with $\mathbb{Q}$, A is open in the subspace topology of $\mathbb{Q}$.

(i) Prove, on the other hand, that A is not open in $\mathbb{R}$.
(ii) What is the closure of A in $\mathbb{R}$ (denoted $\mathrm{cl}_{\mathbb{R}}(A)$)?
(iii) What is the closure of A in $\mathbb{Q}$ (denoted $\mathrm{cl}_{\mathbb{Q}}(A)$)?

Solution.

(i) Suppose for contradiction that A were open in $\mathbb{R}$ and pick $x \in A$. Then there would be some $\varepsilon > 0$ such that $(x-\varepsilon, x+\varepsilon) \subset A \subset \mathbb{Q}$. But any open interval in $\mathbb{R}$ contains infinitely many irrational points, so this is impossible.

(ii) We first note that $A = A_+ \cup A_-$ with $A_+ = \mathbb{Q} \cap (\sqrt{2}, \sqrt{3})$ and $A_- = \mathbb{Q} \cap (-\sqrt{3}, -\sqrt{2})$. Therefore, $\mathrm{cl}_{\mathbb{R}}(A) = \mathrm{cl}_{\mathbb{R}}(A_+) \cup \mathrm{cl}_{\mathbb{R}}(A_-)$. To compute $\mathrm{cl}_{\mathbb{R}}(A_+)$, first suppose that $x \in (\sqrt{2}, \sqrt{3})$ and let V be some open neigborhood of x in $\mathbb{R}$. Then $(\sqrt{2}, \sqrt{3}) \cap V$ is open, so that there is some $\varepsilon > 0$ with $(x-\varepsilon, x+\varepsilon) \subset (\sqrt{2}, \sqrt{3}) \cap V$. Because any open interval in $\mathbb{R}$ contains infinitely many rational numbers we conclude that $A_+ \cap V = \mathbb{Q} \cap (\sqrt{2}, \sqrt{3}) \cap V \neq \emptyset$. We conclude that $(\sqrt{2}, \sqrt{3}) \subset \mathrm{cl}_{\mathbb{R}}(A_+)$. Now, $\{\sqrt{2} + \frac{1}{n}\}_{n\in\mathbb{N}}$ is a sequence (eventually) in $(\sqrt{2}, \sqrt{3})$ that converges to $\sqrt{2}$ and similarly $\{\sqrt{3} - \frac{1}{n}\}_{n\in\mathbb{N}}$

is a sequence (eventually) in $(\sqrt{2}, \sqrt{3})$ converging to $\sqrt{3}$. Therefore, $\sqrt{2}, \sqrt{3} \in \mathrm{cl}_{\mathbb{R}}((\sqrt{2}, \sqrt{3}))$ and we conclude that $[\sqrt{2}, \sqrt{3}] \subset \mathrm{cl}_{\mathbb{R}}(A_+)$. On the other hand, $A_+ \subset [\sqrt{2}, \sqrt{3}]$ and because $[\sqrt{2}, \sqrt{3}]$ is closed in $\mathbb{R}$ we also have $\mathrm{cl}_{\mathbb{R}}(A_+) \subset [\sqrt{2}, \sqrt{3}]$. So, $\mathrm{cl}_{\mathbb{R}}(A_+) = [\sqrt{2}, \sqrt{3}]$.
An entirely analogous argument shows that $\mathrm{cl}_{\mathbb{R}}(A_-) = [-\sqrt{3}, -\sqrt{2}]$. Therefore, we can conclude in summary that $\mathrm{cl}_{\mathbb{R}}(A) = [-\sqrt{3}, -\sqrt{2}] \cup [\sqrt{2}, \sqrt{3}]$.

(iii) Quite generally, we have $\mathrm{cl}_{\mathbb{Q}}(A) = \mathrm{cl}_{\mathbb{R}}(A) \cap \mathbb{Q}$. Therefore,

$$\mathrm{cl}_{\mathbb{Q}}(A) = \mathbb{Q} \cap ([-\sqrt{3}, -\sqrt{2}] \cup [\sqrt{2}, \sqrt{3}]) = \{p \in \mathbb{Q} : 2 \le p^2 \le 3\} = \{p \in \mathbb{Q} : 2 < p^2 < 3\} = A$$

since $\pm\sqrt{2}, \pm\sqrt{3} \notin \mathbb{Q}$.