Math 440 Homework 6

Due Monday, Oct. 16, 2017 by 4 pm

Please remember to write down your name on your assignment.

Please submit your homework to our TA Viktor Kleen, either in his mailbox (in KAP 405) or under the door of his office (KAP 413). You may also e-mail your solutions to Viktor provided:

- you have typed your homework solutions; or
- you are able to produce a very high quality scanned PDF (no photos please!),

1. *Pasting continuous functions together.* Let the topological space X be the union of two closed sets C_1 and C_2 (for instance, $\mathbb{R} = [2, \infty) \cup (-\infty, 3]$). Let Y be another topological space, and consider two maps $f_1 : C_1 \to Y$ and $f_2 : C_2 \to Y$ which are continuous when C_1 and C_2 are endowed with the *subspace topology* Finally, suppose that $f_1(x) = f_2(x)$ for every $x \in C_1 \cap C_2$, so that we can define a map

$$f : X = C_1 \cup C_2 \to Y$$

without ambiguity as

$$f(x) = \begin{cases} f_1(x) & \text{if } x \in C_1 \\ f_2(x) & \text{if } x \in C_2 \end{cases}$$

(a) Show that $f : X \to Y$ is continuous.
Hint: Recall that a function f is continuous if and only if, for every closed subset C of Y, $f^{-1}(C)$ is closed in X).
Hint 2: Let $Y \subset X$ be any closed set, and let $\mathcal{T}_Y$ denote the subspace topology on Y. It will be helpful to use the following fact: if Y is closed in X, then A is closed in Y (with respect to the subspace topology) if and only if A is closed in X.

(b) Show by counterexample that this conclusion may fail if we do not assume that C_1 and C_2 are closed. (*Hint*: one example begins by taking $X = \mathbb{R}$ with its standard topology, C_1 some closed set like $[2, \infty)$, and C_2 its complement $\mathbb{R} - [2, \infty)$; this is open in $\mathbb{R}$ but is not closed. Note that $C_1 \cup C_2 = X$ and $C_1 \cap C_2 = \emptyset$, so the condition that $f_1(x) = f_2(x)$ for every $x \in C_1 \cap C_2$ is empty. Can you now find an example of such an f_1 and f_2 which are both continuous, but so that the associated function f is not continuous?)

(c) Use part (a) to prove that the function

$$f : \mathbb{R} \to \mathbb{R}$$

$$x \mapsto \begin{cases} |x| & x < 2 \\ x^2 - 2 & x \geq 2 \end{cases}$$

is continuous, where $\mathbb{R}$ has its standard topology. *Note*: You may assume that $g(x) = |x|$ and $h(x) = x^2 - 2$ are continuous as functions from $\mathbb{R}$ to $\mathbb{R}$ with its standard topology; indeed $g(x)$ is the Euclidean distance from x to 0, and $h(x)$ is a polynomial.

2. Let (X, d) be any metric space. In class we showed that $\mathcal{B}_d = \{B_d(x, r)\}_{x \in X, r \in (0,\infty)}$ is a basis for a topology on X, and we asserted that the topology $\mathcal{T}_{\mathcal{B}_d}$ generated by $\mathcal{B}_d$ agrees with the underlying topology of the metric space $\mathcal{T}_d$.[1]

   (a) For this question, let $X = \mathbb{R}$. By our discussion in class, the collection $\mathcal{B}_{d_{Eu}} = \{(a, b) = B_{d_{Eu}}(\frac{a+b}{2}, \frac{b-a}{2}) \mid a, b \in \mathbb{R},\ a < b\}$ is a basis for a topology on $\mathbb{R}$, and by above it generates the standard topology $\mathcal{T}_{std} = \mathcal{T}_{d_{Eu}}$. Show that $\mathcal{B}_{\mathbb{Q}} = \{(a, b) | a, b \in \mathbb{Q}, a < b\}$ is another basis for a topology on $\mathbb{R}$, and it generates the same topology.

   (b) Show that $\mathcal{B}_{left} = \{[a, b) | a, b \in \mathbb{R}, a < b\}$ and $\mathcal{B}_{left,\mathbb{Q}} = \{[a, b) | a, b \in \mathbb{Q}, a < b\}$ are also both bases for topologies on $\mathbb{R}$.

   (c) Let $\mathcal{T}_{left}$ denote the topology generated by $\mathcal{B}_{left}$ and $\mathcal{T}_{left,\mathbb{Q}}$ the topology generated by $\mathcal{B}_{left,\mathbb{Q}}$. Note that these topologies are different from the standard topology, because, for example, $[1, 2)$ is never open in the standard topology but is open in both of these topologies. Show that the topologies $\mathcal{T}_{left}$ and $\mathcal{T}_{left,\mathbb{Q}}$ are *not* identical; that is $\mathcal{T}_{left} \neq \mathcal{T}_{left,\mathbb{Q}}$. Is one contained in the other?
   Hint: Consider a set of the form $[a, b)$ with a irrational.

3. (a) Let A, B, C, D be topological spaces, and $f : A \to C$ and $g : B \to D$ two continuous functions. Show that $f \times g : A \times B \to C \times D$, defined by $(a, b) \mapsto (f(a), g(b))$ is continuous too.

   (b) If X is any topological space, show that the *diagonal map* $\Delta : X \to X \times X$ sending $x \mapsto (x, x)$ is continuous. *Note*: As a corollary of (*a*) and (*b*), we note that if $f : X \to Y$ and $g : X \to Z$ are both continuous, then $(f, g) = (f \times g) \circ \Delta : X \to Y \times Z$, sending $x \mapsto (f(x), g(x))$, must be continuous.

4. First, read about the *product* vs. the *box* topologies for products of topological spaces $\prod_{i \in I} X_i$ in Munkres 2.19 (for the former, you will need to look up the notion of a "subbasis."). Here is a direct definition: The box topology is the topology generated by the basis $\{\prod_{i \in I} U_i | U_i \subset X_i \text{ is open}\}$ and the product topology is the topology generated by the basis $\{\prod_{i \in I} U_i | U_i \subset X_i \text{ is open and all but finitely many } U_i \text{ are equal to } X_i\}$, or equivalently $\{\prod_{i \in I} U_i |\ U_i \subset X_i \text{ is open and the set } \{i \in I | U_i \neq X_i\} \text{ is finite.}\}$. These two topologies coincide for a finite product $X_1 \times \cdots \times X_n$, but not for general products.

   Then, solve Munkres section 2.19, number 7.
   Hint: Recall that $\mathbb{R}^\omega$ denotes the set of all sequences of real numbers $\mathbb{R}^\omega = \{\{x_i\}_{i \in \mathbb{N}} | x_i \in \mathbb{R}\}$. Consider for instance the point, e.g., $(5, 5, 5, 5, \ldots,) \in \mathbb{R}^\omega$, and take any open set around it. Does this open set intersect $\mathbb{R}^\infty$ in the two topologies that you're considering?

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[1] Here is a proof: if U is open with respect to the metric (X, d), then for every $x \in U$, some ball centered at x, $B = B_d(x, \delta)$, is contained in U. Since $B \in \mathcal{B}_d$, it follows that U is open in $\mathcal{T}_{\mathcal{B}_d}$. Conversely, say U is open in $\mathcal{T}_{\mathcal{B}_d}$. This means that at any $x \in U$, there is some $B \in \mathcal{B}_d$ containing x contained in U. Without loss of generality, $B = B_d(y, s)$ for some y and s. Now, since we've already shown B is open with respect to the metric topology on (X, d), there exists a ball B' centered at x contained in B. Therefore B' is contained in U too; therefore, since x was arbitrary, U is open with respect to the metric (X, d).