Homework 6

EXERCISE 6.1. Let the topological space X be the union of two closed sets C_1 and C_2. Let Y be another topological space, and consider two maps $f_1 : C_1 \longrightarrow Y$ and $f_2 : C_2 \longrightarrow Y$ which are continuous when C_1 and C_2 are endowed with the subspace topology. Finally, suppose that $f_1(x) = f_2(x)$ for every $x \in C_1 \cap C_2$, so that we can define a map

$$f : X = C_1 \cup C_2 \longrightarrow Y$$

without ambiguity as

$$f(x) = \begin{cases} f_1(x) & \text{if } x \in C_1 \\ f_2(x) & \text{if } x \in C_2. \end{cases}$$

(i) Show that $f : X \longrightarrow Y$ is continuous.
(ii) Show by counterexample that this conclusion may fail if we do not assume that C_1 and C_2 are closed.
(iii) Use part (i) to prove that the function

$$f : \mathbb{R} \longrightarrow \mathbb{R}, \quad x \longmapsto \begin{cases} |x| & \text{if } x < 2 \\ x^2 - 2 & \text{if } x \geq 2 \end{cases}$$

is continuous, where $\mathbb{R}$ has its standard topology.

Solution.

(i) Let $A \subset Y$ be any closed set and consider $f^{-1}(A) \subset X$. Then $f^{-1}(A) \cap C_i = f_i^{-1}(A)$ for $i = 1, 2$: If $x \in f_i^{-1}(A)$ then $x \in C_i$ and $f(x) = f_i(x) \in A$. Therefore, $x \in f^{-1}(A) \cap C_i$. Conversely, if $x \in f^{-1}(A) \cap C_i$ then $f_i(x) = f(x) \in A$ and therefore $x \in f_i^{-1}(A)$.
Now, because $X = C_1 \cup C_2$ we also have $f^{-1}(A) = (f^{-1}(A) \cap C_1) \cup (f^{-1}(A) \cap C_2) = f_1^{-1}(A) \cup f_2^{-1}(A)$. Because f_i is a continuous function $C_i \longrightarrow Y$ for $i = 1, 2$, we know that $f_i^{-1}(A)$ is closed *in* C_i. But we assumed that C_i is closed in X, so in fact $f_i^{-1}(A)$ is closed in X as well. But then $f^{-1}(A) = f_1^{-1}(A) \cup f_2^{-1}(A)$ is a finite union of closed sets and therefore closed itself. Since $A \subset Y$ was an arbitrary closed set we conclude that f is a continuous function $X \longrightarrow Y$.

(ii) Take for example $C_1 = (-\infty, 0) \subset \mathbb{R}$ and $C_2 = [0, \infty) \subset \mathbb{R}$. Let $f_1 : C_1 \longrightarrow \mathbb{R}$ be the constant function $f_1(x) = 0$ and $f_2 : C_2 \longrightarrow \mathbb{R}$ the function with $f_2(x) = 1$. Then both f_1 and f_2, being constant functions, are continuous. However, the function

$$f : \mathbb{R} \longrightarrow \mathbb{R}, \quad x \longmapsto \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases}$$

is not continuous: consider the open set $(0, 2) \subset \mathbb{R}$. Its preimage under f is $f^{-1}((0, 2)) = [0, \infty)$ which is not open in $\mathbb{R}$.

(iii) Note that the way f is defined is not an instance of the procedur in part (i) because $(-\infty, 2)$ is not a closed set in $\mathbb{R}$. However, let's consider the functions $g_1 : (-\infty, 2]$ with $g_1(x) = |x|$ and $g_2 : [2, \infty) \longrightarrow \mathbb{R}$ with $g_2(x) = x^2 - 2$. Both are continuous and for $x \in (-\infty, 2] \cap [2, \infty) = \{2\}$ we have $g_1(2) = 2 = 4 - 2 = g_2(2)$. Therefore, part (i) produces a continuous function $g : \mathbb{R} \longrightarrow \mathbb{R}$ with

$$g(x) = \begin{cases} |x| = g_1(x) & \text{if } x \leq 2 \\ x^2 - 2 = g_2(x) & \text{if } x \geq 2. \end{cases}$$

Now this function actually is equal to f because for every $x \in \mathbb{R}$ we have $f(x) = g(x)$. So f is continuous as well.

Exercise 6.2. Let (X, d) be any metric space. In class we showed that $\mathscr{B}_d = \{B_d(x,r)\}_{x\in X, r>0}$ is a basis for a topology on X, and we asserted that the topology $\mathscr{T}_{\mathscr{B}_d}$ generated by $\mathscr{B}_d$ agrees with the underlying topology of the metric space $\mathscr{T}_d$.

(i) For this question, let $X = \mathbb{R}$. By our discussion in class, the collection $\mathscr{B}_{d_{\mathrm{Eu}}} = \{(a,b) : a, b \in \mathbb{R}, a < b\}$ is a basis for a topology on $\mathbb{R}$, and by the above it generates the standard topology $\mathscr{T}_{d_{\mathrm{Eu}}}$. Show that $\mathscr{B}_{\mathbb{Q}} = \{(a,b) : a, b \in \mathbb{Q}, a < b\}$ is another basis for a topology on $\mathbb{R}$, and it generates the same topology.

(ii) Show that $\mathscr{B}_{\mathrm{left}} = \{[a,b) : a, b \in \mathbb{R}, a < b\}$ and $\mathscr{B}_{\mathrm{left},\mathbb{Q}} = \{[a,b) : a, b \in \mathbb{Q}, a < b\}$ are also both bases for topologies on $\mathbb{R}$.

(iii) Let $\mathscr{T}_{\mathrm{left}}$ denote the topology generated by $\mathscr{B}_{\mathrm{left}}$ and $\mathscr{T}_{\mathrm{left},\mathbb{Q}}$ the topology generated by $\mathscr{B}_{\mathrm{left},\mathbb{Q}}$. Note that these topologies are different from the standard topology , because, for example, $[1,2)$ is never open in the standard topology but is open in both of these topologies. Show that the topologies $\mathscr{T}_{\mathrm{left}}$ and $\mathscr{T}_{\mathrm{left},\mathbb{Q}}$ are *not* identical; that is $\mathscr{T}_{\mathrm{left}} \neq \mathscr{T}_{\mathrm{left},\mathbb{Q}}$. Is one contained in the other?

Solution.

(i) First, $\bigcup_{(a,b)\in\mathscr{B}_{\mathbb{Q}}}(a,b) = \mathbb{R}$: Indeed, for any $x \in \mathbb{R}$, there are rational numbers $a, b \in \mathbb{Q}$ with $a < x < b$, for example $a = \lfloor x \rfloor - 1$ and $b = \lfloor x \rfloor + 1$ work. Furthermore, if $a, b, c, d \in \mathbb{Q}$ then for the intersection of (a,b) and (c,d) we find $(a,b) \cap (c,d) = (\max\{a,c\}, \min\{b,d\}) \in \mathscr{B}_{\mathbb{Q}}$ (or maybe the intersection is empty; but then we don't need to know anything further about it). This is enough to conclude that $\mathscr{B}_{\mathbb{Q}}$ is a basis for a topology on $\mathbb{R}$.
To show that the topology generated by $\mathscr{B}_{\mathbb{Q}}$ is the standard topology, first observe that $\mathscr{B}_{\mathbb{Q}} \subset \mathscr{B}_{d_{\mathrm{Eu}}}$. Therefore, $\mathscr{T}_{\mathscr{B}_{\mathbb{Q}}} \subset \mathscr{T}_{d_{\mathrm{Eu}}}$ as well. Conversely, suppose that $U \in \mathscr{T}_{d_{\mathrm{Eu}}}$. To see that $U \in \mathscr{T}_{\mathscr{B}_{\mathbb{Q}}}$ it will be enough to show that for any $x \in U$ there are rational number a and b with $x \in (a,b) \subset U$. But U is open in the standard topology, so there must be an interval $(c,d) \subset U$ with $x \in (c,d)$. The rational numbers $\mathbb{Q}$ are dense in $\mathbb{R}$ and therefore there is a rational number $a \in (c,x)$ and a rational number $b \in (x,d)$. Then $x \in (a,b)$ and $(a,b) \subset (c,d) \subset U$.

(ii) For $x \in \mathbb{R}$ we have $x \in [\lfloor x \rfloor, \lfloor x \rfloor + 1) \in \mathscr{B}_{\mathrm{left},\mathbb{Q}} \subset \mathscr{B}_{\mathrm{left}}$. Also, for any $a, b, c, d \in \mathbb{R}$ we have $[a,b) \cap [c,d) = [\max\{a,c\}, \min\{b,d\}) \in \mathscr{B}_{\mathrm{left}}$ and if $a, b, c, d \in \mathbb{Q}$ then $[a,b) \cap [c,d) \in \mathscr{B}_{\mathrm{left},\mathbb{Q}}$. This is enough to conclude that $\mathscr{B}_{\mathrm{left}}$ and $\mathscr{B}_{\mathrm{left},\mathbb{Q}}$ are bases for topologies on $\mathbb{R}$.

(iii) Since $\mathscr{B}_{\mathrm{left},\mathbb{Q}} \subset \mathscr{B}_{\mathrm{left}}$, we have $\mathscr{T}_{\mathrm{left},\mathbb{Q}} \subset \mathscr{T}_{\mathrm{left}}$. On the other hand $[\sqrt{2}, 2) \in \mathscr{T}_{\mathrm{left}}$ but $[\sqrt{2}, 2) \notin \mathscr{T}_{\mathrm{left}}$: Otherwise, there would have to be rational numbers a and b with $\sqrt{2} \in [a,b)$ and $[a,b) \subset [\sqrt{2}, 2)$. But $\sqrt{2}$ is irrational, so we would have $a < \sqrt{2}$ and $\sqrt{2} \leq a$ and this is impossible.

Exercise 6.3.

(i) Let A, B, C, and D be topological spaces, and $f : A \longrightarrow C$ and $g : B \longrightarrow D$ two continuous functions. Show that $f \times g : A \times B \longrightarrow C \times D$, defined by $(a,b) \longmapsto (f(a), g(b))$ is continuous too.

(ii) If X is any topological space, show that the *diagonal map* $\Delta : X \longrightarrow X \times X$ sending $x \longmapsto (x,x)$ is continuous.

Solution.

(i) The set $\{U \times V : U \subset C \text{ and } V \subset D \text{ open}\}$ is a basis for the produt topology on $C \times D$. Consequently, to check that $f \times g$ is continuous it will be enough to check that $(f \times g)^{-1}(U \times V)$ is open for all open sets $U \subset C$ and $V \subset D$. But $(f \times g)^{-1}(U \times V) = f^{-1}(U) \times g^{-1}(V)$ and $f^{-1}(U) \subset A$ and $g^{-1}(V) \subset B$ are both open by the continuity of f and g. Therefore $f^{-1}(U) \times g^{-1}(V)$ is open and we conclude that $f \times g$ is continuous.

(ii) Similarly to (i) the set $\{U \times V : U, V \subset X \text{ open}\}$ is a basis for the product topology on $X \times X$. To see that Δ is continuous, it is enough to check that $\Delta^{-1}(U \times V) \subset X$ is open whenever $U, V \subset X$ are open. But $\Delta^{-1}(U \times V) = U \cap V$ is a finite intersection of open sets, so it is open itself.

Exercise 6.4 (Munkres, 2.19.7). Let $\mathbb{R}^\infty$ be the subset of $\mathbb{R}^\omega = \{(x_1, x_2, \ldots) : x_i \in \mathbb{R}\}$ consisting of all sequences that are eventually zero, that is, all sequences $(x_1, x_2, \ldots)$ such that $x_i \neq 0$ for only finitely many values of i. What is the closure of $\mathbb{R}^\infty$ in $\mathbb{R}^\omega$ in the box and product topologies?

Solution. Suppose $x = (x_1, x_2, \dots) \in \mathbb{R}^\omega \setminus \mathbb{R}^\infty$. This means that $x_i \neq 0$ for infinitely many $i \in \mathbb{N}$. Now, define

$$U_i = \begin{cases} \mathbb{R} & \text{if } x_i = 0 \\ \mathbb{R} \setminus \{0\} & \text{if } x_i \neq 0 \end{cases}$$

and observe that each U_i is an open subset of $\mathbb{R}$ containing x_i. Consequently, $U = \prod_i U_i$ is an open subset of $\mathbb{R}^\omega$ in the box topology and $x \in U$. Let $y = (y_1, y_2, \dots) \in U$ and observe that $y_i \neq 0$ whenever $x_i \neq 0$. Hence, $y_i \neq 0$ for infinitely many $i \in \mathbb{N}$ and therefore $y \notin \mathbb{R}^\infty$. This shows that $U \subset \mathbb{R}^\omega \setminus \mathbb{R}^\infty$. Since x was chosen arbitrarily we conclude that $\mathbb{R}^\omega \setminus \mathbb{R}^\infty$ is open in the box topology and $\text{cl}(\mathbb{R}^\infty) = \mathbb{R}^\infty$ in the box topology.

On the other hand, suppose that $U \subset \mathbb{R}^\omega$ is an open set in the product topology and $x = (x_1, x_2, \dots) \in U$. Then U contains an open neighborhood of x of the form $V = V_1 \times V_2 \times \dots$ with each V_i open in $\mathbb{R}$ and there is some $N > 0$ such that $V_i = \mathbb{R}$ for $i \geq N$. Set $y = (x_1, x_2, \dots, x_N, 0, 0, \dots) \in V \subset U$. Then $y \in \mathbb{R}^\infty$ and consequently $U \cap \mathbb{R}^\infty \neq \emptyset$. This means that any nonempty open subset of $\mathbb{R}^\omega$ in the product topology intersects $\mathbb{R}^\infty$ and therefore $\text{cl}(\mathbb{R}^\infty) = \mathbb{R}^\omega$ in the product topology.