Math 440 Homework 7

Due Friday, Oct. 27, 2017 by 4 pm

Please remember to write down your name on your assignment.

Please submit your homework to our TA Viktor Kleen, either in his mailbox (in KAP 405) or under the door of his office (KAP 413). You may also e-mail your solutions to Viktor provided:

- you have typed your homework solutions; or
- you are able to produce a very high quality scanned PDF (no photos please!),

1. Prove, using the methods developed in class that the following subset of $\mathbb{R}^2$
$$X = \{(x,y)|4x^2 + 3y^2 = 1\} \cup \{(x,0)|x \in \mathbb{R}, x \geq \frac{1}{2}\} \cup \{(-\frac{1}{2} - t, t)|t \in \mathbb{R}, t \geq 0\},$$
equipped with the subspace topology (for the standard topology on $\mathbb{R}^2$), is connected.

2. Let $f : X \to Y$ be a continuous function between the topological spaces X and Y. The *graph* of f is the subset
$$\Gamma_f = \{(x,y) \in X \times Y | y = f(x)\}$$
of the product $X \times Y$. Endow $X \times Y$ with the product topology, and $\Gamma_f \subset X \times Y$ with its subspace topology.
   (a) Show that the function $g : X \to \Gamma_f$ defined by $g(x) = (x, f(x))$ is a homeomorphism. (you will need to show it is continuous first)

   (b) Assume in addition that the topological space Y is Hausdorff. Show that the graph Γ_f is a closed subset of $X \times Y$.

   (c) Show that the hypotheses that Y is Hausdorff is necessary in the previous question. That is, give an example of a continuous function f whose graph Γ_f is not closed in $X \times Y$. (*Hint*: Y should necessarily be non-Hausdorff, so cannot for example be a subset of $\mathbb{R}$ with the subspace topology).

3. Let $Y \subset X$ be a connected subspace of a topological space. Let Z be any subset containing Y and contained in the closure of Y; so
$$Y \subset Z \subset \text{cl}(Y).$$
(in particular, Z includes every element of Y and some subset of the collection of *limit points* of Y — $Z = Y$ and $Z = \text{cl}(Y)$ are special cases).

   Show, assuming that Y is connected, that Z is connected too.

4. Let Y once more be a connected subspace of a topological space X. By the previous problem, $\text{cl}(Y)$ is connected too. Are the following spaces connected too? Prove or disprove by counter example.

(a) ∂Y.
(b) $\mathrm{int}(Y)$.

5. Show that $(0,1)$ and $(0,1]$ (thought of as subspaces of $\mathbb{R}$ with its standard topology) are *not* homeomorphic. *Hint*: what happens if you remove a point from each of these spaces? It would be helpful to prove that
   (i) the property of being *connected* is preserved under homeomorphism[1], and
   (ii) if $f : X \to Y$ is a homeomorphism, and $p \in X$ any point, then $f|_{X-\{p\}} : X - \{p\} \to Y - \{f(p)\}$ is a homeomorphism too, where $X - \{p\}$ and $Y - \{f(p)\}$ are equipped with the subspace topology.

6. Suppose $X = [0,1]$ equipped with its subspace topology, and let $f : X \to X$ be a continuous function. Show that f has a *fixed point*; that is, a point $x \in [0,1]$ with $f(x) = x$. Is this also true if $X = [0,1)$ or $(0,1)$? *Hint*: If there are no such points, then for any $x \in [0,1]$ either $f(x) > x$ or $f(x) < x$. Can you use this to construct a separation of X?

7. Show that if U is an *open* connected subspace of $\mathbb{R}^2$ with its standard topology, then U is path connected (this notion will be defined in class on Friday 10/20). *Hint*: show that given $x_0 \in U$, the set P_{x_0} of points that can be joined to x_0 by a path in U is both open and closed in U. If so, by connectedness of U, P_{x_0} is either empty or all of U!

[1] Proof sketch: Say $f : X \to Y$ is a homeomorphism. If $X = A \cup B$ is a separation of X; then using the fact f is a homeomorphism, one can deduce that $Y = f(A) \cup f(B)$ is a separation of Y. Conversely, by applying f^{-1}, one shows a separation of Y induces a separation of X. This shows that X is not connected iff Y is not connected, equivalently X is connected iff Y is.