Homework 7

Exercise 7.1. Prove, using the methods developed in class that the following subset of $\mathbb{R}^2$

$$X = \{(x,y) : 4x^2 + 3y^2 = 1\} \cup \{(x,0) : x \in \mathbb{R}, x \geq \tfrac{1}{2}\} \cup \{(-\tfrac{1}{2} - t, t) : t \in \mathbb{R}, t \geq 0\},$$

equipped with the subspace topology, is connected.
Solution. Write $X = X_1 \cup X_2 \cup X_3$ with

$$\begin{aligned} X_1 &= \{(x,y) : 4x^2 + 3y^2 = 1\} \\ X_2 &= \{(x,0) : x \in \mathbb{R}, x \geq \tfrac{1}{2}\} \\ X_3 &= \{(-\tfrac{1}{2} - t, t) : t \in \mathbb{R}, t \geq 0\}. \end{aligned}$$

We first show that X_1, X_2, and X_3 are connected. For this, let $f\colon [0, 2\pi] \longrightarrow \mathbb{R}^2$ be defined by

$$f(t) = (\tfrac{1}{2}\cos t, \tfrac{1}{\sqrt{3}}\sin t).$$

Then f is continuous since each of its component functions is and $f([0, 2\pi]) = X_1$. Therefore, X_1 is connected since intervals in $\mathbb{R}$ are connected. Now, $X_2 = [\frac{1}{2}, \infty) \times \{0\}$ is homeomorphic to the interval $[\frac{1}{2}, 0) \subset \mathbb{R}$ which is connected. Therefore, X_2 is connected as well. Furthermore, let $g\colon [0, \infty) \longrightarrow \mathbb{R}^2$ be the function defined by $g(t) = (-\frac{1}{2} - t, t)$. This function is continuous and $f([0, \infty)) = X_3$. Hence, X_3 is connected as well.

Note that $(\frac{1}{2}, 0) \in X_1 \cap X_2$ and $(-\frac{1}{2}, 0) \in X_1 \cap X_3$. This means that $X_1 \cap X_2 \neq \emptyset$ and therefore $X_1 \cup X_2$ is connected. Also, $(X_1 \cup X_2) \cap X_3 \supset X_1 \cap X_3 \neq \emptyset$, so $X_1 \cup X_2 \cup X_3 = X$ is connected as well.

Exercise 7.2. Let $f\colon X \longrightarrow Y$ be a continuous function between the topological spaces X and Y. The *graph* of f is the subset

$$\Gamma_f = \{(x,y) \in X \times Y : y = f(x)\}$$

of the product $X \times Y$. Endow $X \times Y$ with the product topology, and $\Gamma_f \subset X \times Y$ with its subspace topology.

(i) Show that the function $g\colon X \longrightarrow \Gamma_f$ defined by $g(x) = (x, f(x))$ is a homeomorphism.
(ii) Assume in addition that the topological space Y is Hausdorff. Show that the graph Γ_f is a closed subset of $X \times Y$.
(iii) Show that the hypotheses that Y is Hausdorff is necessary in the previous question. That is, give a example of a continuous function f whose graph Γ_f is not closed in $X \times Y$.

Solution.

(i) First, since $f\colon X \longrightarrow Y$ and the identity function $X \longrightarrow X$ are both continuous and $\Gamma_f \subset X \times Y$ carries the subspace topology, by the definition of the product topology, $g\colon X \longrightarrow \Gamma_f$ is a continuous function. Consider the projection $\pi\colon X \times Y \longrightarrow X$ and restrict it to a function $\pi\colon \Gamma_f \longrightarrow X$. Again this is continuous and furthermore

$$\pi(g(x)) = \pi(x, f(x)) = x \quad \text{and} \quad g(\pi(x, f(x))) = g(x) = (x, f(x))$$

for all $(x, f(x)) \in \Gamma_f$. We conclude that $g\colon X \longrightarrow \Gamma_f$ is a homeomorphism with inverse $\pi\colon \Gamma_f \longrightarrow X$.

(ii) We show that $\Gamma_f^c = X \times Y \setminus \Gamma_f$ is open. Suppose that $(x,y) \in \Gamma_f^c$, that is, $y \neq f(x) \in Y$. Since Y is Hausdorff there are disjoint open sets U and V containing y and $f(x)$ respectively. Because f is continuous the preimage $f^{-1}(V) \subset X$ is open. Consequently. $f^{-1}(V) \times U \subset X \times Y$ is open. Furthermore, if $z \in f^{-1}(V)$, then $f(z) \in V$ and therefore $f(z) \notin U$, that is $(z, f(z)) \notin f^{-1}(V) \times U$. This means that $f^{-1}(V) \times U \subset \Gamma_f^c$. We conclude that Γ_f^c is open and therefore Γ_f is closed.

(iii) Let $X = \{a, b\}$ be endowed with the indiscrete topology and consider the identity function $\mathrm{id}\colon X \longrightarrow X$. Since the product topology on $X \times X$ is also the indiscrete topology and $\Gamma_{\mathrm{id}} \neq \emptyset, X \times X$ this gives the required counterexample.

EXERCISE 7.3. Let $Y \subset X$ be a connected subspace of a topological space. Let Z be any subset containing Y and contained in the closure of Y; so

$$Y \subset Z \subset \mathrm{cl}(Y).$$

Show, assuming that Y is connected, that Z is connected too.

Solution. Suppose for contradiction that there were open subsets $A, B \subset X$ such that $Z \subset A \cup B$, $Z \cap A \neq \emptyset$, $Z \cap B \neq \emptyset$ and $A \cap B = \emptyset$. Take $x \in Z \cap A$. Then $x \in \mathrm{cl}(Y)$ and therefore $Y \cap A \neq \emptyset$ because A is open in X and $x \in A$. Similarly, $Y \cap B \neq \emptyset$. But this is impossible since Y is connected.

EXERCISE 7.4. Let Y once more be a connected subspace of a topological space X. By the previous problem, $\mathrm{cl}(Y)$ is connected, too. Are the following spaces connected too? Prove or disprove by counterexample.

(i) The boundary ∂Y.
(ii) The interior $\mathrm{int}(Y)$.

Solution.

(i) The interval $[0, 1] \subset \mathbb{R}$ is closed, connected and $\mathrm{int}([0, 1]) = (0, 1)$. Therefore, $\partial[0, 1] = \{0, 1\}$ and $\{0, 1\}$ is not connected.
(ii) Consider the closed set $X = [0, 1]^2 \cup [-1, 0]^2 \subset \mathbb{R}^2$. Both $[0, 1]^2$ and $[-1, 0]^2$ are connected and $[0, 1]^2 \cap [-1, 0]^2 = \{(0, 0)\} \neq \emptyset$. Hence, X is connected as well. But $\mathrm{int}(X) = (0, 1)^2 \cup (-1, 0)^2$ is a disjoint union of two nonempty open sets and therefore is disconnected.

EXERCISE 7.5. Show that $(0, 1)$ and $(0, 1]$, thought of as subspaces of $\mathbb{R}$ with its standard topology, are not homeomorphic.

Solution. First, if $f : X \longrightarrow Y$ is a homeomorphism and $x \in X$. Then $f|_{X \setminus \{x\}} : X \setminus \{x\} \longrightarrow Y \setminus \{f(x)\}$ is continuous and similarly $f^{-1}|_{Y \setminus \{f(x)\}} : Y \setminus \{f(x)\} \longrightarrow X \setminus \{x\}$ is continuous. Furthermore, these two maps are inverses of each other, so we conclude that $X \setminus \{x\}$ and $Y \setminus \{f(x)\}$ are homeomorphic as well.

Now, suppose there were a homeomorphism $f : (0, 1] \longrightarrow (0, 1)$. Then we would conclude that $(0, 1)$ and $(0, 1) \setminus \{f(1)\}$ are homeomorphic as well. But $0 < f(1) < 1$ and therefore $(0, 1) \setminus \{f(1)\} = (0, f(1)) \cup (f(1), 1)$ is disconnected while $(0, 1)$ is connected. So these spaces cannot be homeomorphic and we conclude that $(0, 1]$ and $(0, 1)$ cannot be homeomorphic either.

EXERCISE 7.6. Suppose $X = [0, 1]$ equipped with its subspace topology, and let $f : X \longrightarrow X$ be a continuous function. Show that f has a *fixed point*; that is, a point $x \in [0, 1]$ with $f(x) = x$. Is this also true if $X = [0, 1)$ or $(0, 1)$?

Solution. Suppose that f does not have a fixed point. Then for any $x \in X$ we would need to have either $f(x) > x$ or $f(x) < x$. That is to say that $X = \{x \in X : f(x) > x\} \cup \{x \in X : f(x) < x\} = A \cup B$. Observe that the function $g : X \longrightarrow \mathbb{R}$ with $g(x) = f(x) - x$ is continuous and $A = g^{-1}((0, \infty)) = A$ and $B = g^{-1}((-\infty, 0))$. Hence, A and B are open and disjoint. Also, $f(0) \in (0, 1]$ and $f(1) \in [0, 1)$ and therefore $f(0) \in A$ and $f(1) \in B$. But this would imply that $[0, 1]$ is disconnected which is false. So f must have had a fixed point after all.

For $X = [0, 1)$ and $X = (0, 1)$ not every continuous function $f : X \longrightarrow X$ must have a fixed point. For example, suppose $f(x) = \frac{1}{2}(1 - x) + x$. Then $f(x) = x$ if and only if $x = 1$ and therefore f doesn't have a fixed point in X.

EXERCISE 7.7. Show that if U is an *open* connected subspace of $\mathbb{R}^2$ with its standard topology, then U is path connected.

Solution. Suppose $x_0 \in U$ and let P be the set of all points $x \in U$ that can be connected to x_0 by a continuous path. Suppose $x \in P$. Then there is some $\varepsilon > 0$ such that $B(x, \varepsilon) \subset U$ because U is open. Let $y \in B(x, \varepsilon)$ and suppose that $\gamma : [0, 1] \longrightarrow U$ is a continuous path with $\gamma(0) = x_0$ and $\gamma(1) = x$. Consider the map $\gamma' : [0, 1] \longrightarrow U$ with

$$\gamma'(t) = \begin{cases} \gamma(2t) & t < \frac{1}{2} \\ (2 - 2t)x + (2t - 1)y & t \geq \frac{1}{2}. \end{cases}$$

First, this is well defined since $\gamma(2t) \in U$ for all t and

$$\|\gamma'(t) - x\| = \|(1-2t)(x-y)\| = |1-2t|\|x-y\| < |1-2t|\varepsilon < \varepsilon$$

for $\frac{1}{2} \leq t < 1$. Furthermore, γ' is continuous and in summary we have constructed a path from x_0 to y. Hence, $y \in P$ and since $y \in B(x, \varepsilon)$ was arbitrary we conclude that $B(x, \varepsilon) \subset P$. Consequently, P is a nonempty open subset of U.

On the other hand, suppose $x \in U \setminus P$. Again, there is some $\varepsilon > 0$ with $B(x, \varepsilon) \subset U$. Let $y \in B(x, \varepsilon)$ and suppose for contradiction that $y \in P$. Then there would be a path $\gamma \colon [0, 1] \longrightarrow U$ from x_0 and y. But then we would again have a continuous map $\gamma' \colon [0, 1] \longrightarrow U$ with

$$\gamma'(t) = \begin{cases} \gamma(2t) & t < \frac{1}{2} \\ (2-2t)y + (2t-1)x & t \geq \frac{1}{2}. \end{cases}$$

That is, there would also be a path connecting x_0 and x. But $x \in U \setminus P$, so this is impossible. Consequently, $B(x, \varepsilon) \subset U \setminus P$ and therefore $U \setminus P$ is open.

But now we have $U = P \cup (U \setminus P)$ with $P \neq \emptyset$ and P and $U \setminus P$ both open. Because U is connected we must have $U \setminus P = \emptyset$, i. e. $U = P$. But this means that U is path connected.