Math 440 Homework 8

Due Wednesday, Nov. 8, 2017 by 4 pm

Please remember to write down your name on your assignment.

Please submit your homework to our TA Viktor Kleen, either in his mailbox (in KAP 405) or under the door of his office (KAP 413). You may also e-mail your solutions to Viktor provided:

- you have typed your homework solutions; or
- you are able to produce a very high quality scanned PDF (no photos please!),

1. *Connected components and path-connected components of a space.* Read Munkres §3.25, up to the statement and proof of Theorem 25.2. This section concerns in particular, the notion of a *path component.*

   (a) π_0 *of a space.* Given a topological space X, define

   $$\pi_0(X) = \{\text{the set of path-connected components of } X\}.$$

   In other words, $\pi_0(X)$ is the quotient of the underlying set X by the equivalence relation $x \sim y$ iff there is a path in X from x to y.[1]

   Prove that $\pi_0(X)$ is an invariant of X, via the following approach: Prove that a continuous map $f : X \to Y$ induces a well-defined map $f_* : \pi_0(X) \to \pi_0(Y)$, and that for $id_X : X \to X$ the induced map is $(id_X)_* = id_{\pi_0(X)} : \pi_0(X) \to \pi_0(X)$. Conclude that a homeomorphism $f : X \to Y$ induces a bijection $f_* : \pi_0(X) \to \pi_0(Y)$.

   Note: The notation π_0 comes from the fact that there is an invariant of topological spaces called $\pi_n(X)$ for any $n \geq 0$, which often go by the name *homotopy groups.* (this is the "zeroth homotopy group")

   (b) Show, using the invariant π_0, that the following two subsets of $\mathbb{R}^2$, equipped with the quotient topology, are not homeomorphic:

   $$A = \mathbb{R}^2 - \partial B_{d_{Eu}}((0,0),1) = \{(x,y) \in \mathbb{R}^2 | x^2 + y^2 \neq 1\}$$

   $$B = \mathbb{R}^2 - (\partial B_{d_{Eu}}((0,0),\frac{1}{2}) \cup \partial B_{d_{Eu}}((0,0),1)) = \{(x,y) \in \mathbb{R}^2 | x^2 + y^2 \notin \{\frac{1}{4}, 1\}\}.$$

   In words, A is $\mathbb{R}^2$ minus a circle (of radius 1 centered at 0), and B is $\mathbb{R}^2$ minus a pair of nested circles (both centered at 0, of radius $\frac{1}{2}$ and 1 respectively). *Hint*: compute $\pi_0(A)$ and $\pi_0(B)$ and prove there is no bijection between these two sets.

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[1] Why is this an equivalence relation? It is *reflexive* because there is always a path from x to x, the constant path $[0,1] \to X$ sending $t \mapsto x$ for all t. It is *symmetric*, because whenever $x \sim y$, so there is a path γ from x to y, by reversing the path $\bar{\gamma}(t) = \gamma(1-t)$, I obtain a path from y to x and hence $y \sim x$. It is *transitive*, because whenever there is a path γ_1 from x to y and a path γ_2 from y to z, I can *concatenate* these paths to get a path from x to z which first follows γ_1 then γ_2: $\bar{\gamma} = \gamma_1(2t)$ if $t \leq \frac{1}{2}$ and $\gamma_2(2t-1)$ if $t \geq \frac{1}{2}$.

2. Show by example that the tube lemma fails if Y is not compact. That is, find a pair of spaces X, Y with Y *not compact*, a point $x \in X$, and an open neighborhood $N \subset X \times Y$, containing the slice $\{x\} \times Y$, such that for any open set $W \subset X$ containing x, N does not contain the tube $W \times Y$.

3. (a) Let X be a topological space. Show that any finite union of compact subsets of X is again compact.

   (b) *A spiky torus and a point.* Let A denote the following subset of $\mathbb{R}^3$, equipped with the subspace topology:

$$A = \{(x,y,z) \in \mathbb{R}^3 \mid (4-\sqrt{x^2+y^2})^2+z^2 = 2^2\} \cup \{(x,0,z) \in \mathbb{R}^3 \mid x \in \{-2,2\}, 2 \le z \le 4\} \cup \{(0,0,6)\}.$$

   Attempt to draw A using a method of your choice (you are welcome to use a computer tool): visually A should look like the surface of a donut (hence the word "torus"), with two attached line segments, as well as an extra point not on the donut or line segments. Then show, using a method of your choice, that A is compact.

4. *Distance to a closed set.* Let d denote any metric on $\mathbb{R}^n$ inducing the standard topology, and let $A \subset \mathbb{R}^n$ be a closed, *not necessarily bounded* set (so A need not be compact), and $p \in \mathbb{R}^n$. Show that the function

$$d(p,-) : A \to \mathbb{R}$$

achieves a minimum on A; that is, there exists $x \in A$ with $d(p,x) \le d(p,y)$ for all $y \in A$.

Note: We call the resulting minimal distance from p to any point on A the *distance from p to A*, denoted $d(p,A)$.

Hint: Pick any $x_0 \in A$, and set $R = d(p,x_0)$. Argue that one can apply the extreme value theorem to $d(p,-)$ restricted to the subset $A \cap \bar{B}_d(p,R)$ (you'll need to justify this subset is non-empty and compact). Why is the resulting minimum of $d(p,-) : A \cap \bar{B}_d(p,R) \to \mathbb{R}$ also a minimum for $d(p,-) : A \to \mathbb{R}$?