Homework 8

Exercise 8.1.

(i) Given a topological space X, define

$$\pi_0(X) = \{\text{path–connected components of } X\}.$$

In other words, $\pi_0(X)$ is the quotient of the underlying set X by the equivalence relation $x \sim y$ iff there is a path in X from x to y.

Prove that $\pi_0(X)$ is an invariant of X, via the following approach: Prove that a continuous map $f\colon X \longrightarrow Y$ induces a well-defined map $f_*\colon \pi_0(X) \longrightarrow \pi_0(Y)$, and that for $\mathrm{id}_X\colon X \longrightarrow X$ the induced map is $(\mathrm{id}_X)_* = \mathrm{id}_{\pi_0(X)}\colon \pi_0(X) \longrightarrow \pi_0(X)$. Conclude that a homeomorphism $f\colon X \longrightarrow Y$ induces a bijection $f_*\colon \pi_0(X) \longrightarrow \pi_0(Y)$.

(ii) Show, using the invariant π_0, that the following two subsets of $\mathbb{R}^2$, equipped with the subset topology, are not homeomorphic:

$$A = \mathbb{R}^2 \setminus \partial B((0,0),1) = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \neq 1\}$$
$$B = \mathbb{R}^2 \setminus (\partial B((0,0),\tfrac{1}{2}) \cup B((0,0),1)) = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \notin \{\tfrac{1}{4}, 1\}\}.$$

Solution.

(i) Suppose $f\colon X \longrightarrow Y$ is a continuous map and $[x] \in \pi_0(X)$ is the path–connected component of X containing $x \in X$. We claim that that the path–connected component of $f(x) \in Y$ does not depend on our choice of x; in other words, $[f(x)] = [f(x')] \in \pi_0(Y)$ whenever $[x] = [x'] \in \pi_0(X)$. Indeed, suppose that x and x' are connected by a path $\gamma\colon [0,1] \longrightarrow X$, say $\gamma(0) = x$ and $\gamma(1) = x'$. Then $f \circ \gamma\colon [0,1] \longrightarrow Y$ is a path in Y with $(f \circ \gamma)(0) = f(x)$ and $(f \circ \gamma)(1) = f(x')$. Therefore $[f(x)] = [f(x')] \in \pi_0(Y)$. Consequently, we have a well-defined map

$$f_*\colon \pi_0(X) \longrightarrow \pi_0(Y), \quad [x] \longmapsto [f(x)].$$

We immediately see that $(\mathrm{id}_X)_* = \mathrm{id}_{\pi_0(X)}$: just compute $(\mathrm{id}_X)_*([x]) = [\mathrm{id}_X(x)] = [x]$.
Furthermore, if $f\colon X \longrightarrow Y$ and $g\colon Y \longrightarrow Z$ are both continuous, then we compute

$$(g \circ f)_*([x]) = [(g \circ f)(x)] = [g(f(x))] = g_*([f(x)]) = g_*(f_*([x])) = (g_* \circ f_*)([x])$$

for all $[x] \in \pi_0(X)$. This just means that $(g \circ f)_* = g_* \circ f_*$. These properties of the construction $(\_)_*$ allow us to give an easy prove that a homeomorphism $f\colon X \longrightarrow Y$ induces a bijection $\pi_0(X) \longrightarrow \pi_0(Y)$. Namely, suppose that $f\colon X \longrightarrow Y$ admits a continuous inverse $g\colon Y \longrightarrow X$. Then $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$. So we see that

$$g_* \circ f_* = (g \circ f)_* = (\mathrm{id}_X)_* = \mathrm{id}_{\pi_0(X)} \quad \text{and} \quad f_* \circ g_* = (f \circ g)_* = (\mathrm{id}_Y)_* = \mathrm{id}_{\pi_0(Y)}.$$

In other words, $f_*\colon \pi_0(X) \longrightarrow \pi_0(Y)$ has an inverse map $g_*\colon \pi_0(Y) \longrightarrow \pi_0(X)$ and therefore must be a bijection.

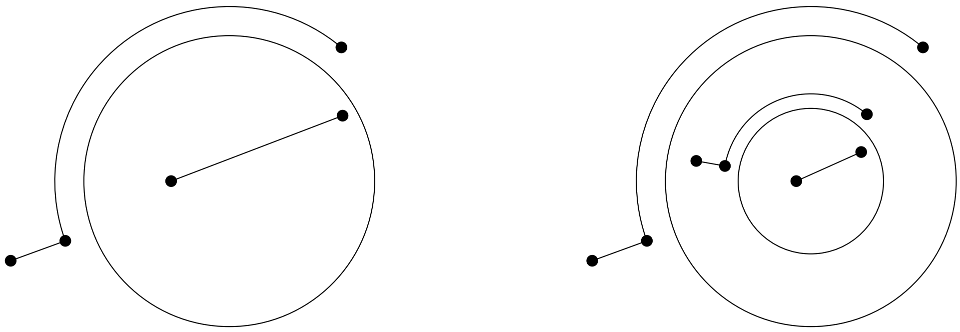
(ii) Let's first compute $\pi_0(A)$. Note that there is a decomposition $A = U_1 \cup U_2$ into nonempty disjoint open sets

$$U_1 = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1\} = B_1(0)$$

and

$$U_2 = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 > 1\}.$$

Both U_1 and U_2 are path–connected. For $U_1 = B_1(0)$ we know this already, so suppose $x, y \in U_2$, say $|x| \le |y|$. Then $|x| \cdot y/|y|$ and x both lie on the circle with radius $|x|$, so they are connected by a continuous path in U_2. The point $|x| \cdot y/|y|$ lies on a straight ray from the origin passing through y and because the distance from the origin to $|x| \cdot y/|y|$ is strictly bigger than 1 we see that $|x| \cdot y/|y|$ and y are connected by a continuous path in U_2. We conclude that $\pi_0(A) = \{U_1, U_2\}$ is a set of cardinality 2.



Similarly, there is a decomposition $B = U_1 \cup U_2 \cup U_3$ into nonempty disjoint open sets

$$U_1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < \tfrac{1}{4}\},$$
$$U_2 = \{(x, y) \in \mathbb{R}^2 : \tfrac{1}{4} < x^2 + y^2 < 1\},$$

and

$$U_3 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 > 1\}.$$

Analogous to the previous argument we see that U_1, U_2 and U_3 are path–connected and conclude that $\pi_0(B) = \{U_1, U_2, U_3\}$ has cardinality 3. If there were a homeomorphism $A \longrightarrow B$, then by part (i) it would induce a bijection $\pi_0(A) \longrightarrow \pi_0(B)$ and $\pi_0(B)$ would have to have cardinality 2. Therefore, there can be no such homeomorphism.

Exercise 8.2. Show by example that the tube lemma fails if Y is not compact. That is, find a pair of spaces X, Y with Y not compact, a point $x \in X$, and an open neighborhood $N \subset X \times Y$ containing the slice $\{x\} \times Y$, such that for any open set $W \subset X$ containing x, the neighborhood N does not contain the tube $W \times Y$.
Solution. For example, consider the open set

$$N = \{(x, y) \in \mathbb{R} \times \mathbb{R}_{>0} : |x| < \tfrac{1}{y}\}$$

and observe that $\{0\} \times \mathbb{R}_{>0} \subset N$. However, if W is an open subset of $\mathbb{R}$ containing 0 then there is some $\varepsilon > 0$ such that $(-2\varepsilon, 2\varepsilon) \subset W$. But $(\varepsilon, \frac{1}{\varepsilon}) \notin N$ even though $(\varepsilon, \frac{1}{\varepsilon}) \in W \times \mathbb{R}_{>0}$.

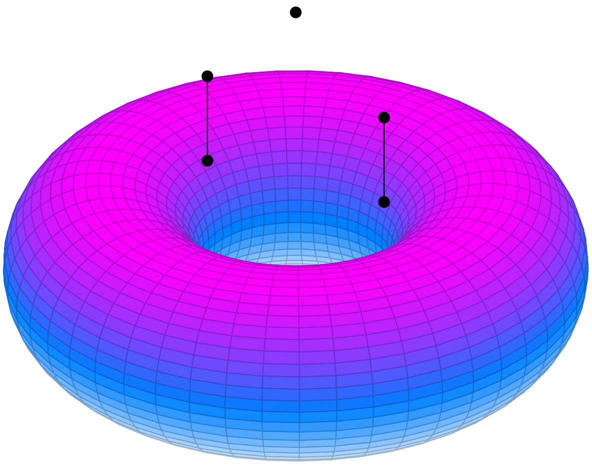
Exercise 8.3.
(i) Let X be a topological space. Show that any finite union of compact subsets of X is again compact.
(ii) Let A denite the following subset of $\mathbb{R}^3$, equipped with the subspace topology:

$$A = \left\{(x, y, z) : \left(4 - \sqrt{x^2 + y^2}\right)^2 + z^2 = 4\right\} \cup \{(x, 0, z) : x \in \{-2, 2\}, 2 \leq z \leq 4\} \cup \{(0, 0, 6)\}.$$

Attempt to draw A using a method of your choice. Then show, using a method of your choice, that A is compact.

Solution.
(i) Suppose $C_1, \dots, C_n \subset X$ are compact subsets of X and write $C = C_1 \cup \cdots \cup C_n$. Suppose $(U_i)_{i \in I}$ is a family of open subsets of X such that $C \subset \bigcup_{i \in I} U_i$. Then for each $1 \leq k \leq n$ we have $C_k \subset \bigcup_{i \in I} U_i$ and, because C_k is compact, there is a finite subset $I_k \subset I$ such that $C_k \subset \bigcup_{i \in I_k} U_i$. But then $J = I_1 \cup \cdots \cup I_n$ is still a finite subset of I and furthermore $C = C_1 \cup \cdots \cup C_n \subset \bigcup_{i \in J} U_i$. Since the open cover $(U_i)_{i \in I}$ was arbitrary, C must be compact.

(ii) Write $A = A_1 \cup A_2 \cup A_3$ for the decomposition in the question. Here is a picture of the situation:



Note that the torus A_1 is the preimage $f^{-1}(4)$ for the continuous function

$$f\colon \mathbb{R}^3 \longrightarrow \mathbb{R}, \quad (x,y,z) \longmapsto \left(4 - \sqrt{x^2+y^2}\right)^2 + z^2.$$

Hence, A_1 is closed. The set A_2 is the union of $\{-2\} \times \{0\} \times [2,4]$ and $\{2\} \times \{0\} \times [2,4]$. But $[2,4]$ is compact and $\{x\} \times \{0\} \times [2,4]$ is homeomorphic to $[2,4]$ for any x. In particular, A_2 is closed as well. Since $\mathbb{R}^3$ is Hausdorff, $A_3 = \{(0,0,6)\}$ is closed as well. So A is a finite union of closed sets and therefore closed itself. Now, A is also bounded, $A \subset B_{10}(0)$ for example. Hence, Heine–Borel implies that A is compact.

Exercise 8.4. Let d denote any metric on $\mathbb{R}^n$ inducing the standard topology, and let $A \subset \mathbb{R}^n$ be a closed, not necessarily bounded set, and $p \in \mathbb{R}^n$. Show that function

$$d(p,-)\colon A \longrightarrow \mathbb{R}$$

achieves a minimum on A; that is, there exists some $x \in A$ with $d(p,x) \le d(p,y)$ for all $y \in A$.

Solution. Suppose[1] that $x \in A$ and let $R = d(p,x)$. The closed ball $\overline{B}_d(p,R)$ around p is compact since d induces the standard topology on $\mathbb{R}^n$. Because A is closed, the intersection $A \cap \overline{B}_d(p,R)$ is a closed subset of the compact set $\overline{B}_d(p,R)$ and therefore compact itself. Since the function $d(p,-)|_{A\cap\overline{B}_d(p,R)}\colon A \cap \overline{B}_d(p,R) \longrightarrow \mathbb{R}$ is continuous it must attain a minimum, say $d(p,x_0) \le d(p,x)$ for all $x \in A \cap \overline{B}_d(p,R)$. But now, if $x \in A$ is arbitrary, then either $d(p,x) > R$, in which case $d(p,x_0) \le R < d(p,x)$, or $d(p,x) \in \overline{B}_d(p,R)$, in which case $d(p,x_0) \le d(p,x)$ by our construction of x_0. Hence, $d(p,-)\colon A \longrightarrow \mathbb{R}$ attains a minimum at $x_0 \in A$.

[1] If $A = \emptyset$ then the statement is false but uninteresting.