Math 440 Homework 9—Half weight

Due Thursday, Nov. 30, 2017 by 4 pm

Please remember to write down your name on your assignment.

Please submit your homework to our TA Viktor Kleen, either in his mailbox (in KAP 405) or under the door of his office (KAP 413). You may also e-mail your solutions to Viktor provided:

- you have typed your homework solutions; or
- you are able to produce a very high quality scanned PDF (no photos please!),

1. *The line with two origins.* In this exercise, our goal is to show that the *Hausdorff* condition is not necessarily preserved by taking quotient spaces. We will do this by constructing a topological space, the *line with two origins*, which is (homeomorphic to) a quotient of a Hausdorff space yet not Hausdorff.

   (a) Let $Y = \mathbb{R} \cup \{0'\}$ be the real numbers union an additional point, which we call $0'$ ("the second origin"). Equip Y with the following topology: any open subset of $\mathbb{R} - \{0\}$ is open when thought of as a subset of Y, and if $U \subset \mathbb{R}$ is any open subset of $\mathbb{R}$ containing 0, then U, $U - \{0\} \cup \{0'\}$ and $U \cup \{0'\}$ are all open in Y. In other words $U \subset Y$ is open iff one of the two conditions hold:
   - U contains neither 0 nor $0'$, and U is open in $\mathbb{R}$; or
   - U contains one or both of 0 and $0'$, and the result of replacing these elements by the single element 0, $(U - \{0, 0'\}) \cup \{0\}$, is an open subset of $\mathbb{R}$.

   You may assume this defines a topology; we call the resulting topological space Y the *line with two origins.* If it is helpful you may assume that the topology on Y is generated by the following basis $\mathcal{B}$:

   $$\mathcal{B} = \{(a,b) \subset \mathbb{R} - \{0\} \subset Y \mid\} \cup \{(-r,r) \subset \mathbb{R} \subset Y\} \cup \{(-r,0) \cup \{0'\} \cup \{0,r\}\}.$$

   in words, $\mathcal{B}$ consists of all open intervals avoiding 0 entirely, open intervals containing 0, and open intervals containing 0 with 0 replaced by $0'$.

   Prove that $(Y, \mathcal{T}_Y)$ is not Hausdorff. *Hint*: prove that 0 and $0'$ cannot be separated from each other.

   (b) Let $X = \{(x,i) \in \mathbb{R}^2 | i \in \{0,1\}\}$ be the union of two lines $y = 0$ (the x-axis) and $y = 1$ topology from $\mathbb{R}^2$. Note that X is Hausdorff (as more generally, any subspace of a Hausdorff space is Hausdorff[1]).

   Let $\bar{X}$ be the following partition of X: $\bar{X}$ consists of the sets $\{(x,0),(x,1)\}$ for each $x \in \mathbb{R} - \{0\}$, as well as $\{(0,0)\}$ and $\{(0,1)\}$. (we might say, as we did in class, that $\bar{X}$ is the quotient of X by the equivalence relation "generated by" $(x,0) \sim (x,1)$ for all $x \neq 0$).

[1] Proof: Let Z be a Hausdorff space, A any subspace, and $p, q \in A$ any two non-equal points in A. Since Z is Hausdorff, there exist disjoint open neighborhoods around p and q in Z, call them U and V. Then note that $U \cap A$ and $V \cap A$ are open disjoint neighborhoods in the subspace topology on A containing p and q respectively. It follows, since p and q were arbitrary, that A is Hausdorff.

Equip $\bar{X}$ with its quotient topology (induced by X and the natural map $p : X \to \bar{X}$). Prove that $\bar{X}$ is homeomorphic to Y, the line with two origins described in (a). Hence, $\bar{X}$ is not Hausdorff.
Hint: there is a bijection $\bar{f} : \bar{X} \to Y$ which sends the element $\{(x,0),(x,1)\}$ of $\bar{X}$ to $x \in \mathbb{R} \subset Y$, sends the element $\{(0,0)\} \in \bar{X}$ to 0, and sends $\{(0,1)\}$ to $0'$; why is it continuous and a bijection? And why is its inverse continuous?

2. Let $X = \mathbb{R}^2 - \{0\}$ be the plane minus the origin, with the usual (subspace) topology. Let $\bar{X}$ be the partition of X consisting of the sets $\{(2^n x, 2^n y) | n \in \mathbb{Z}\}$. Namely, two points are in the same set of the partition if and only if one is obtained from the other one by multiplication by a power of 2.

   Let $p : X \to \bar{X}$ be the quotient map, and endow $\bar{X}$ with the quotient topology induced by X and p.

   (a) Consider the map $f : X \to S^1 \times S^1 \subset \mathbb{R}^2 \times \mathbb{R}^2$ defined by

$$f(x,y) = \left(\left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}} \right), \left(\cos\left(2\pi \frac{\log\sqrt{x^2+y^2}}{\log 2} \right), \sin\left(2\pi \frac{\log\sqrt{x^2+y^2}}{\log 2} \right) \right) \right).$$

   Show that f induces a continuous bijection $\bar{f} : \bar{X} \to S^1 \times S^1$ for which $f = \bar{f} \circ p$ (you may assume without proof that the usual functions, including $\sqrt{-}$, cos, sin, etc. are continuous real-valued functions).

   (b) Show that $\bar{X}$ is compact. Possible hint: Find a compact subset $K \subset X$ such that $p(K) = \bar{X}$.

   (c) Show that $\bar{f}$ is a homeomorphism *Hint*: $\bar{X}$ is compact. Is $S^1 \times S^1$ Hausdorff?