Homework 9

Exercise 9.1. In this exercise, our goal is to show that the Hausdorff condition is not necessarily preserved by taking quotient spaces. We will do this by constructing a topological space, the *line with two origins*, which is homeomorphic to a quotient of a Hausdorff space yet not Hausdorff itself.

(i) Let $Y = \mathbb{R} \cup \{0'\}$ be the real numbers union an additional point, which we call $0'$ (the "second origin"). Equip Y with the following topology: any open subset of $\mathbb{R} \setminus \{0\}$ is open when thought of as a subset of Y, and if $U \subset \mathbb{R}$ is any open subset of $\mathbb{R}$ containing 0, then U, $U \setminus \{0\} \cup \{0'\}$ and $U \cup \{0'\}$ are all open in Y. In other words, $U \subset Y$ is open if and only if one of the following two conditions hold:
- U contains neither 0 nor $0'$, and U is open in $\mathbb{R}$; or
- U contains one or both of 0 and $0'$, and the result of replacing these elements by the single element 0, $(U \setminus \{0, 0'\}) \cup \{0\}$, is an open subset of $\mathbb{R}$.

You may assume that this defines a topology; we call the resulting topological space Y the *line with two origins*. If it is helpful you may assume that the topology on Y is generated by the following basis $\mathscr{B}$:

$$\mathscr{B} = \{(a, b) \subset \mathbb{R} \setminus \{0\} \subset Y\} \cup \{(-r, r) \subset \mathbb{R} \subset Y\} \cup \{(-r, 0) \cup \{0'\} \cup (0, r) \subset Y\},$$

in other words, $\mathscr{B}$ consists of all open intervals avoiding 0 entirely, open intervals containing 0, and open intervals containing 0 with 0 replaced by $0'$.

Prove that $(Y, \mathscr{T}_Y)$ is not Hausdorff.

(ii) Let $X = \{(x, i) \in \mathbb{R}^2 : i \in \{0, 1\}$ be the union of the two lines $y = 0$ and $y = 1$ with the induced topology from $\mathbb{R}^2$. Note that X is Hausdorff, as more generally any subspace of a Hausdorff space is Hausdorff. Let $\overline{X}$ be the following partition of X: the set $\overline{X}$ consists of the sets $\{(x, 0), (x, 1)\}$ for each $x \in \mathbb{R} \setminus \{0\}$, as well as $\{(0, 0)\}$ and $\{(1, 0)\}$. We might say, as we did in class, that $\overline{X}$ is the quotient of X by the equivalence relation "generated by" $(x, 0) \sim (x, 1)$ for all $x \neq 0$.

Equip $\overline{X}$ with its quotient topology induced by X and the natural map $p\colon X \longrightarrow \overline{X}$. Prove that $\overline{X}$ is homeomorphic to Y, the line with two origins described in (i). Hence, $\overline{X}$ is not Hausdorff.

Solution.

(i) Suppose U is any open neighborhood of 0 in Y. By the definition of $\mathscr{T}_Y$ it follows that U must contain a small open interval $(-\varepsilon, \varepsilon) \subset \mathbb{R}$. Now, if V is an open neighborhood of $0'$, then there similarly must be some $\varepsilon' > 0$ such that $(-\varepsilon', 0) \cup \{0'\} \cup (0, \varepsilon') \subset V$. But if $r = \min\{\varepsilon, \varepsilon'\} > 0$ then $(0, r) \subset U \cap V$. In particular, $U \cap V \neq \emptyset$. This means that 0 and $0'$ cannot be separated by open sets in Y.

(ii) First consider the map

$$f\colon X \longrightarrow Y, \qquad \begin{cases} (x, i) \longmapsto x & \text{whenever } x \neq 0 \\ (0, 0) \longmapsto 0 \\ (0, 1) \longmapsto 0'. \end{cases}$$

This map is in fact continuous: Suppose $U \subset Y$ is open. If $U \subset \mathbb{R} \setminus \{0\}$ and U is open in $\mathbb{R}$, then $f^{-1}(U) = U \times \{0, 1\} \subset X$ is open in X. If U contains 0 or $0'$ then by definition $V = (U \setminus \{0'\}) \cup \{0\}$ is an open subset of $\mathbb{R}$. The preimage $f^{-1}(U)$ is one of $V \times \{0, 1\}$, $(V \times \{0, 1\}) \setminus \{(0, 0)\}$ or $(V \times \{0, 1\}) \setminus \{(0, 1)\}$. Since both $\{(0, 0)\}$ and $\{(0, 1)\}$ are closed in $\mathbb{R}^2$ all of these sets are open subsets of X and therefore $f^{-1}(U)$ is open in all cases.

Notice that $f(x, 0) = f(x, 1)$ for $x \neq 0$. This means that f respects the equivalence relation defining $\overline{X}$ and there is a unique continuous map $\overline{f}\colon \overline{X} \longrightarrow Y$ with $\overline{f} \circ p = f$.

In the other direction, define

$$\overline{g}\colon Y \longrightarrow \overline{X}, \qquad \begin{cases} x \longmapsto \{(x, 0), (x, 1)\} & \text{whenever } x \neq 0 \text{ and } x \neq 0' \\ 0 \longmapsto \{(0, 0)\} \\ 0' \longmapsto \{(0, 1)\}. \end{cases}$$

Then one sees immediately that $\overline{f} \circ \overline{g} = \mathrm{id}_Y$ and $\overline{g} \circ \overline{f} = \mathrm{id}_{\overline{X}}$, so we only need to check that $\overline{g}$ is continuous. Suppose $U \subset \overline{X}$ is open. First, assume that $\{(0,0)\}, \{(0,1)\} \notin U$. Then $p^{-1}(U) = V \times \{0,1\}$ for some open set $V \subset \mathbb{R} \setminus \{0\}$ and $\overline{g}^{-1}(U) = V$ is open in Y. If $\{(0,0)\} \in U$ or $\{(0,1)\} \in U$ then $p^{-1}(U)$ is of the form $V \times \{0,1\}$, $(V \times \{0,1\}) \setminus \{(0,0)\}$ or $(V \times \{0,1\}) \setminus \{(0,1)\}$ for some open set $V \subset \mathbb{R}$ containing 0. This means that $\overline{g}^{-1}(U)$ is one of $V \cup \{0'\}$, $(V \setminus \{0\}) \cup \{0'\}$ or V respectively. All of these sets are open in Y and we can conclude that $\overline{g}$ is continuous.
In summary, we have found a continuous map $\overline{f} \colon \overline{X} \longrightarrow Y$ with a continuous inverse $\overline{g} \colon Y \longrightarrow \overline{X}$ and there $\overline{X}$ and Y are homeomorphic.

Exercise 9.2. Let $X = \mathbb{R}^2 \setminus \{0\}$ be the plane minus the origin, with the usual (subspace) topology. Let $\overline{X}$ be the partition of X consisting of sets $\{(2^n x, 2^n y) : n \in \mathbb{Z}\}$. Namely, two points are in the same set of the partition if and only if one is obtained from the other one by multiplication by a power of 2.

Let $p \colon X \longrightarrow \overline{X}$ be the quotient map, and endow $\overline{X}$ with the quotient topology induced by X and p.

(i) Consider the map $f \colon X \longrightarrow S^1 \times S^1 \subset \mathbb{R}^2 \times \mathbb{R}^2$ defined by

$$f(x,y) = \left(\frac{(x,y)}{\sqrt{x^2+y^2}}, \left(\cos\left(2\pi \frac{\log\sqrt{x^2+y^2}}{\log 2}\right), \sin\left(2\pi \frac{\log\sqrt{x^2+y^2}}{\log 2}\right) \right) \right).$$

Show that f induces a continuous bijection $\overline{f} \colon \overline{X} \longrightarrow S^1 \times S^1$ for which $f = \overline{f} \circ p$.

(ii) Show that $\overline{X}$ is compact.

(iii) Show that $\overline{f}$ is a homeomorphism.

Solution.

(i) We first note that $f \colon X \longrightarrow S^1 \times S^1$ is surjective: suppose $((a,b),(c,d)) \in S^1 \times S^1$. Then there is some $\phi \in \mathbb{R}$ with $\cos\phi = c$ and $\sin\phi = d$. Because $\log \colon \mathbb{R}_{>0} \longrightarrow \mathbb{R}$ is surjective there is some $r > 0$ with $\phi = 2\pi \log r \log 2$. Set $(x,y) = r \cdot (a,b)$ and compute:

$$\frac{(x,y)}{\sqrt{x^2+y^2}} = \frac{r(a,b)}{r\sqrt{a^2+b^2}} = (a,b)$$
$$\frac{2\pi}{\log 2} \log\sqrt{x^2+y^2} = \frac{2\pi}{\log 2} \log(r\sqrt{a^2+b^2}) = 2\pi \frac{\log r}{\log 2} = \phi.$$

Consequently, $f(x,y) = ((a,b),(c,d))$.
Next, assume that $(x', y') = 2^n(x,y) \in X$ for some $n \in \mathbb{Z}$. Then

$$\frac{(x',y')}{\sqrt{x'^2+y'^2}} = \frac{2^n(x,y)}{\sqrt{2^{2n}x^2+2^{2n}y^2}} = \frac{(x,y)}{\sqrt{x^2+y^2}}$$
$$\frac{2\pi}{\log 2} \log\sqrt{x'^2+y'^2} = \frac{2\pi}{\log 2} \log\left(2^n \sqrt{x^2+y^2}\right) = \frac{2\pi}{\log 2} \log 2^n + \frac{2\pi}{\log 2} \log\sqrt{x^2+y^2} =$$
$$= 2\pi n + \frac{2\pi}{\log 2} \log\sqrt{x^2+y^2}.$$

But cos and sin are periodic with period 2π. Hence, $f(x',y') = f(x,y)$. Conversely, suppose that $f(x',y') = f(x,y)$ for $(x,y), (x',y') \in X$. Then we have

$$\cos\left(\tfrac{2\pi}{\log 2} \log\sqrt{x^2+y^2}\right) = \cos\left(\tfrac{2\pi}{\log 2} \log\sqrt{x'^2+y'^2}\right)$$

and

$$\sin\left(\tfrac{2\pi}{\log 2} \log\sqrt{x^2+y^2}\right) = \sin\left(\tfrac{2\pi}{\log 2} \log\sqrt{x'^2+y'^2}\right).$$

The function $\phi \longmapsto (\cos\phi, \sin\phi)$ is periodic with period 2π and injective on $[0, 2\pi)$. Therefore there must be some $n \in \mathbb{Z}$ with

$$\frac{2\pi}{\log 2} \log \sqrt{x^2 + y^2} + 2\pi n = \frac{2\pi}{\log 2} \log \sqrt{x'^2 + y'^2}.$$

Performing the previous calculations backwards, this means

$$\frac{2\pi}{\log 2} \log \sqrt{2^{2n}x^2 + 2^{2n}y^2} = \frac{2\pi}{\log 2} \log \sqrt{x'^2 + y'^2}.$$

The logarithm and the square root are injective functions $\mathbb{R}_{>0} \longrightarrow \mathbb{R}$, so this let's us conclude that $2^{2n}(x^2+y^2) = x'^2+y'^2$. But now, writing $f(a,b) = (f_1(a,b), f_2(a,b))$ and remembering our assumption that $f(x,y) = f(x',y')$ we can compute

$$\begin{aligned} 2^n(x,y) &= 2^n\sqrt{x^2+y^2}\frac{(x,y)}{\sqrt{x^2+y^2}} = \sqrt{2^{2n}x^2+2^{2n}y^2}\frac{(x,y)}{\sqrt{x^2+y^2}} = \sqrt{x'^2+y'^2}f_1(x,y) = \\ &= \sqrt{x'^2+y'^2}f_1(x',y') = \sqrt{x'^2+y'^2}\frac{(x',y')}{\sqrt{x'^2+y'^2}} = \\ &= (x',y'). \end{aligned}$$

In summary, we have $f(x,y) = f(x',y')$ for $(x,y),(x',y') \in X$ if and only if there is some $n \in \mathbb{Z}$ with $2^n(x,y) = (x',y')$. This implies that $f\colon X \longrightarrow S^1 \times S^1$ descends to a bijection $\overline{f}\colon \overline{X} \longrightarrow S^1 \times S^1$ with $f = \overline{f} \circ p$. Furthermore, this map $\overline{f}$ is continuous because f was continuous and $\overline{X}$ carries the quotient topology.

(ii) Consider the closed annulus $A = \overline{B}(0,2) \smallsetminus B(0,1) \subset \mathbb{R}^2 \smallsetminus \{0\}$. This set is both closed and bounded and therefore compact by Heine–Borel. So it will suffice to show that $p(A) = \overline{X}$ in order to conclude that $\overline{X}$ is compact. For this, suppose that $(x,y) \in \mathbb{R}^2 \smallsetminus \{0\}$ and write $r = \sqrt{x^2+y^2}$. Then there is some integer $n \in \mathbb{Z}$ such that $2^n \leq r \leq 2^{n+1}$. But this means that $2^{-n}(x,y) \in A$ and $p(x,y) = p(2^{-n}(x,y))$. Consequently, for any $p(x,y) \in \overline{X}$ there is $(x',y') \in A$ such that $p(x',y') = p(x,y)$. In other words, $p(A) = \overline{X}$.

(iii) We have a continuous bijection $\overline{f}\colon \overline{X} \longrightarrow S^1 \times S^1$ from the compact space $\overline{X}$ to the Hausdorff space $S^1 \times S^1$. Any such map must be a homeomorphism.