Math 440 Midterm Exam

Wednesday, October 4, 2017, 11:00-11:50am.

Instructions. Answer the following problems carefully and completely. You must show all of your work, stating any result that you are using (unless it is otherwise clear), in order to receive full credit. You are welcome to use all results from the book or class unless otherwise stated, though no references, paper or digital, are permitted. Write your solutions in the provided space; indicate clearly if any solutions are continued on another side or on scratch paper. Please return this examination, along with any scratch paper used.

If on a multi-part problem you cannot do a part (e.g., part (a)), you may still assume it is true for subsequent parts (e.g., part (b) or (c)) if it is helpful.

Partial credit will be awarded for conceptually correct proofs that are missing some details or justification for steps.

There are 4 problems, worth a total of 80 points (20 points each). You will have 50 minutes to complete the exam.

Name:

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USC ID number:

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1. (20 points total) Let (X, d) be a metric space.

(a) (12 points) Prove that the interior of the closed ball $\bar{B}(x, r)$ always contains the open ball $B(x, r)$; that is prove that there is an inclusion

$$B(x, r) \subset \operatorname{int}(\bar{B}(x, r)).$$

(b) (8 points) Does the reverse inclusion hold? That is, is it always true that $\operatorname{int}(\bar{B}(x, r)) \subset B(x, r)$? If so, prove it; if not, provide a counterexample with justification.

2. (20 points total)

(a) (6 points) Give a definition of the *subspace topology* for a subset A of a topological space $(X, \mathcal{T}_X)$.

(b) (7 points) Let A denote the subset of $(\mathbb{R}, \mathcal{T}_{standard})$ given by $A = [0, 1) \cup \{2\}$ equipped with its subspace topology.

Prove that the subset $V = [\frac{1}{2}, 1) \subset A$ is closed in A.

(c) (7 points) Prove on the other hand that V is not closed in $\mathbb{R}$, or equivalently that $\mathbb{R} - V$ is not open in $\mathbb{R}$.

3. (20 points total)

(10 points) Prove that if X, Y, and Z are topological spaces, and $f : X \to Y$ and $g : Y \to Z$ are continuous functions, then the composition $g \circ f$ is continuous too.

(b) (10 points) Let $X = \frac{1}{2}\mathbb{Z}$ denote the half integers (so $X = \{\ldots, -1, -\frac{1}{2}, 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots\}$, or more concisely $X = \{\frac{n}{2} | n \in \mathbb{Z}\}$). Equip X with the structure of a metric space via the metric

$$d(x, y) = |x - y|.$$

(you may take for granted that d indeed defines a metric). Prove that the induced topology $\mathcal{T}_d$ on X is the *discrete topology.*

4. (20 points total)

(a) (10 points) Show by example that an arbitrary intersection of open sets in a topological space need not remain open.

(b) (10 points) Show that $U = \{(x, y) \in \mathbb{R}^2 | 2x^2y - xy^3 \notin \mathbb{Z}\}$ is an open subset of $\mathbb{R}^2$ with its standard topology.