Math 440 Midterm Exam — Solutions

Wednesday, October 4, 2017, 11:00-11:50am.

1. (20 points total) Let (X, d) be a metric space.

(a) (12 points) Prove that the interior of the closed ball $\bar{B}(x, r)$ always contains the open ball $B(x, r)$; that is prove that there is an inclusion

$$B(x, r) \subset \text{int}(\bar{B}(x, r)).$$

Solution. In any metric space the open ball $B(x, r)$ is an open set (we proved this in class). Furthermore, $B(x, r) \subset \bar{B}(x, r)$ because for any $y \in X$, $d(x, y) < r$ ($y \in B(x, r)$) implies that $d(x, y) \leq r$ ($y \in \bar{B}(x, r)$). But $\text{int}(\bar{B}(x, r))$ is the union of all open sets U that are contained in $\bar{B}(x, r)$. Since we have seen $B(x, r)$ is one of these open sets we conclude that $B(x, r) \subset \text{int}(\bar{B}(x, r))$.

(b) (8 points) Does the reverse inclusion hold? That is, is it always true that $\text{int}(\bar{B}(x, r)) \subset B(x, r)$? If so, prove it; if not, provide a counterexample with justification.

Solution. The reverse inclusion doesn't always hold. Consider a set X with more than two elements and equip it with the discrete metric d. Let $x \in X$. Then consider the ball $B(x, 1)$. We have $B(x, 1) = \{y \in X : d(x, y) < 1\} = \{x\}$ and on the other hand $\bar{B}(x, 1) = \{y \in X : d(x, y) \leq 1\} = X$. Therefore, since $X \subset X$ is open, $\bar{B}(x, 1)$ is open, so it equals its interior: $\text{int}(\bar{B}(x, 1)) = \bar{B}(x, 1) = X$. But $X \not\subset \{x\}$ because X has more than two elements.

2. (20 points total)

(a) (6 points) Give a definition of the *subspace topology* for a subset A of a topological space $(X, \mathcal{T}_X)$. *Solution.* The subspace topology $\mathcal{T}_A$ is given by the formula

$$\mathcal{T}_A = \{U \cap A : U \in \mathcal{T}_X\}.$$

(b) (7 points) Let A denote the subset of $(\mathbb{R}, \mathcal{T}_{standard})$ given by $A = [0,1) \cup \{2\}$ equipped with its subspace topology.

Prove that the subset $V = [\frac{1}{2}, 1) \subset A$ is closed in A.

Solution. By a theorem proved in class we need to find a set $W \subset \mathbb{R}$ which is closed in $\mathbb{R}$ such that $V = W \cap A$. In fact, we can take $W = [\frac{1}{2}, 1]$. This being a closed interval it is a closed set in $\mathbb{R}$ and $W \cap A = V$.

(c) (7 points) Prove on the other hand that V is not closed in $\mathbb{R}$, or equivalently that $\mathbb{R} - V$ is not open in $\mathbb{R}$.

Solution. If V were closed in $\mathbb{R}$, then it would have to contain the limit of any convergent sequence in V. But $\{1 - \frac{1}{n}\}_{n=2}^{\infty}$ converges to 1 as a sequence of real numbers and $1 - \frac{1}{n} \in V$ for $n \geq 2$. So, if V were closed in $\mathbb{R}$, then we would need to have $1 \in V$ which is not the case. Therefore V cannot be closed in $\mathbb{R}$.

3. (20 points total)

(a) (10 points) Prove that if X, Y, and Z are topological spaces, and $f : X \to Y$ and $g : Y \to Z$ are continuous functions, then the composition $g \circ f$ is continuous too.

Solution. Let $U \subset Z$ be any open set. Since g is continuous, by definition the preimage of U under g, $g^{-1}(U)$ must be an open subset of Y. Next, since f is continuous, by definition the preimage of the open set $g^{-1}(U)$ under f, $f^{-1}g^{-1}(U)$, must be an open subset of X. Now, note that $f^{-1}g^{-1}(U) = (g \circ f)^{-1}(U)$ (by comparing definitions), so we have shown that $(g \circ f)^{-1}(U)$ is open, for any open set $U \subset Z$. It follows by definition that $g \circ f$ is continuous.

(b) (10 points) Let $X = \frac{1}{2}\mathbb{Z}$ denote the half integers (so $X = \{\ldots, -1, -\frac{1}{2}, 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots\}$, or more concisely $X = \{\frac{n}{2} | n \in \mathbb{Z}\}$). Equip X with the structure of a metric space via the metric

$$d(x, y) = |x - y|.$$

(you may take for granted that d indeed defines a metric). Prove that the induced topology $\mathcal{T}_d$ on X is the *discrete topology.*

Solution. To show $\mathcal{T}_d$ is the discrete topology, we need to show that any subset of X is open (hence that $\mathcal{T}_d = \mathcal{P}(X)$. So, let $A \subset X$ be an arbitrary subset, and $p = \frac{k}{2} \in A$ any element of A (where $k \in \mathbb{Z}$). Now, note that with respect to the above metric,

$$B_d(p, \frac{1}{4}) = \{p\},$$

that is there are no other half integers, other than p itself, which are distance $< \frac{1}{4}$ from p. Hence,

$$B_d(p, \frac{1}{4}) \subset A.$$

Since $p \in A$ was arbitrary, we have produced, for any $p \in A$, an open ball around p contained in A. Thus, A is open. Since A was arbitrary, it follows any subset of X is open as desired.

4. (20 points total)

(a) (10 points) Show by example that an arbitrary intersection of open sets in a topological space need not remain open.

Solution. There are many possible solutions; here is one option. Let $\mathbb{R}$ denote the real numbers with its standard topology, and consider the indexed family of sets $\{U_n = (-\frac{1}{n}, \frac{1}{n})\}_{n \in \mathbb{N}}$. Then each $U_n = (-\frac{1}{n}, \frac{1}{n})$ is open because it's an open interval, yet the intersection of all U_n's is:

$$\cap_{i \in \mathbb{N}} U_i = \{0\},$$

which is not open in $\mathbb{R}$ with its standard topology.

(b) (10 points) Show that $U = \{(x,y) \in \mathbb{R}^2 | 2x^2y - xy^3 \notin \mathbb{Z}\}$ is an open subset of $\mathbb{R}^2$ with its standard topology.

Solution.

Denoting by $f : \mathbb{R}^2 \to \mathbb{R}$ the function $f(x,y) = 2x^2y - xy^3$, note that U can be written as follows: $U = \{(x,y)|f(x,y) \notin \mathbb{Z}\} = \{(x,y)|f(x,y) \in \mathbb{R} - \mathbb{Z}\} = f^{-1}(\mathbb{R} - \mathbb{Z})$. If we show that (i) that f is continuous and (ii) that $\mathbb{R} - \mathbb{Z} \subset \mathbb{R}$ is open, then it will follow that $U = f^{-1}(\mathbb{R} - \mathbb{Z})$ is open too, because the preimage of any open subset under a continuous function remains open (by the topological definition of continuity).

For point (i): the function $f : (x,y) \mapsto 2x^2y - xy^3$ is continuous as a map from $\mathbb{R}^2$ to $\mathbb{R}$ (both with their standard topologies), as it is polynomial in the components of $\mathbb{R}^2$, and we proved such functions are continuous in class.

For point (ii): $\mathbb{R} - \mathbb{Z}$ is an open subset of $\mathbb{R}$ with its standard topology, because it can be written as the (infinite) union of open intervals $W = \bigcup_{i \in \mathbb{Z}} (i, i+1) = \cdots (-1,0) \cup (0,1) \cup (1,2) \cup \cdots$, and arbitrary unions of open sets remain open.