Math 440 Project Assignment

1. Overview

The goal of your project assignment is to explore an aspect of topology beyond the topics covered in class. It will be necessary to use the tools and properties of topological spaces that we have developed, such as: compactness, connectedness, the notion of a continuous function and homeomorphisms, constructions of topological spaces, etc.

You will work with groups of 2-3 students. As part of this assignment, you will write a short **paper** (4-5 pages), and you will give a **20 minute presentation**.

2. Grading

The final project will be worth **75 points total**, split between a paper (4-6 pages maximum) and presentation (20 minute group presentation) component:

- **10 points**: *Outline submission (due 11/10).* Requirements: a paragraph stating your plan, followed by 1 or 2 page outline detailing your paper plan: sections (ideally 3-4), mention of examples you plan to study, results you will state, results you will prove or sketch, and motivation you will discuss; in the order it appears. Your grade will be based only on completeness/reasonable effort.
- **10 points**: *First draft submission (due 11/20).* You do not need to exactly follow your outline, in case you decided to make changes. Your grade for this part will be based on completion; did you make a reasonable effort to fill out all of the sections you proposed in your outline (suitably adjusted in case you made changes)? It is ok if some specific Lemmas, etc. are not fleshed out, as long as the paper is structurally fleshed out.
- **30 points**: *Final draft submission (4-6 pages maximum, due 12/1).* Here you will be graded on the quality of your submission, according to roughly 4 categories:
  - *Content*: Is a reasonable breadth of content covered, with examples?
  - *Mathematical writing*: are proofs written in understandable English; were common "errors in mathematical writing" avoided? (see e.g., `http://www.math.uconn.edu/~kconrad/math216/mathwriting.pdf`)
  - *Exposition*: is there a sensible overview at the beginning of what is going to happen in the document? Are there transitions between results? Explanations for why intermediate technical results are used? Motivation for why one is doing various things? Did your group judiciously choose what to leave in and what to leave out? Are examples given to explain difficult concepts?
  - *Organization*: Are different parts of content broken up into sensible sections/subsections as needed? Are proofs of results split into reasonable Lemmas, Propositions, etc.?
- **25 points**: *20 min group presentation.* Here you will be graded on *organization* (both of your talk and board work), *exposition* (as above), and *content.* Note: *all members of the group should be part of the presentation, at least for a couple minutes each.*

Note: All members of a group will receive the same score.

3. Timeline/Due Dates

- **Tuesday, October 31 at 1 pm**: **selecting your group and project (or rather, rank your top three options)**. The deadline for selecting your group and projects is Tuesday, October 31. At least one person from each group should e-mail me (`sheel.ganatra@usc.edu`) by then with your group, and your top three project choices. I will respond by Tuesday at 5 pm with your project assignments.
- **Outline**: A proposed project outline is due to me (`sheel.ganatra@usc.edu`) and Viktor (`kleen@usc.edu`) on **Friday, November 10 by 4 pm**. The goal of this outline is for you to flesh out your project plan and proposed references, in bullet point or outline form, and clarify any confusions you might have with me, about what is ok to cover and what is ok to skip. (After this submission, you will receive feedback from me as to whether the outline seems like a reasonable plan for your paper or whether you may wish to consider adding/removing sections).
- **First draft**: The first draft is due **Monday, November 20 at 10 am** *to both Viktor and me* by e-mail. (After this submission, you will receive feedback from Viktor on your draft).
- **Final draft**: The final draft is due **Friday, December 1 at 4 pm** to Viktor and me.
- **Presentation**: You will give a *20 minute* lecture/presentation on your topic, jointly with your group (all members must say something). Final presentations will occur on the last 4 days of class: **November 20, November 27, November 29, and December 1**.

For either submission, you can e-mail your solutions if you would like.

4. Typsetting

It is required that you type your paper using software of your choice. You are *strongly encouraged* (but it is not mandatory) to use LaTeX to write your paper; if you have not learned LaTeX yet, now is an excellent time to do so. To help you along, some LaTeX templates will be provided on the course website.

5. Groups

As mentioned, you will work in groups of 2-3 students. You are welcome to choose your own group, but must submit to me the names of all group members. We are aiming for no more than 8 groups total.

The deadline of submitting the names of all group members to me is this coming Tuesday, Oct. 31.

6. On expository writing

The aim of the written portion of the assignment is for you to practice writing mathematical *exposition*; it is important that your writing be precise yet understandable to someone else in your class with a comparable background. *Understandability* means several things:

- From a mechanical standpoint, you should avoid the use of any mathematical shorthand symbols like $\forall$ and $\exists$; intead write "for all" and "there exists." Always write in complete sentences, and explain conclusions to arguments.
- From a writing standpoint, you may choose to first give conceptual explanations of a fact before giving a rigorous proof. Or going even further, at times you may choose to omit a certain (inessential) proof, but only give a conceptual explanation of it.

7. On presentations

Think carefully about how you want to structure your class time. What are the most important ideas you are trying to convey; what is less important? What results are worth stating but completely omitting the proof of, so that you can get to other key arguments? In case you run out of time, what is okay to skip? *How will you split time among the entire group?*

I would suggest you give one or two practice talks amongst your groups (or to another group); in particular, develop familiarity with using the chalkboard. How does your board look from the back of class?

8. Project ideas

You are welcome to choose one of the following list of projects, or come up with another project with instructor approval. Broadly, the idea is to explore an area with a collection of definitions, examples (with proof), and main results (with at least partial proofs/proof explanations—complete proofs if they are not too long).

Below are some project ideas, and broad suggestions for what details to include:

Note: You are not required to exactly follow the suggestions; feel free to deviate as desired. Also note that project (6) requires understanding the fundamental group, a notion we will define towards the end of class. If you are interested in this direction, you will need to familiarize yourself a little bit with the fundamental before the end of class; I would be happy to help with this.

(1) **Metrization theorems**: When is a topological space $(X, \mathcal{T}_X)$ metrizable? We have seen *necessary conditions*: a X had better be Hausdorff (T2) (and also T1) to have any hope of being metrizable. But these conditions are not *sufficient*: it is not true that if X is Hausdorff, then X must be metrizable. (find an example!).

The goal of this project is to study the metrizability question: specifically, give *sufficient*, or better, *necessary* and *sufficient* conditions for metrizability of a topological space. The relevant possible results are called *Urysohn's metrization theorem* (which is sufficient) and/or the *Nagata-Smirnov theorem* (which is necessary and sufficient) and/or the *Smirnov metrization theorem* (which is necessary and sufficient, and follows from Nagata-Smirnov); these are both described in your textbook (in §4.34, §6.40 and §6.42 respectively). You do not need to describe all of these, and if you describe more than one, you do not need to give complete details for all of them (but should focus on giving more details for at least one of the Theorems, and whatever definitions are necessary to state the other Theorems). Complete proofs are not necessary, but some summary of the proof and relevant definitions are. Examples are also helpful; you may for instance pick one or two of the exercises

(2) **Manifold theory**: A (topological) n-manifold is, roughly, a topological space which is locally homeomorphic to an open subset of $\mathbb{R}^n$. Manifolds are fundamental objects in geometry and topology, and appear frequently in other disciplines as well. There are three possible projects here:

   (a) **An introduction to manifolds and their classification.** Give the definition of an n-manifold, prove basic properties of manifolds, and produce examples (with proof) of some compact connected 1 and 2-dimensional manifolds (in dimension 1, the circle, in dimension 2, the sphere, surfaces of genus g, and real projective space). State (without proof) the classification of 1 and 2-dimensional manifolds up to homeomorphism, and possibly give a few words (paragraph at most) of intuition about why these classifications are true.

   (b) **Manifolds and their embeddings**. Give the definition and some basic examples of an n-manifold, and prove basic properties of manifolds. Define an *embedding* of manifolds $M^m \hookrightarrow N^n$, and give some basic examples. Sketch a proof that any compact manifold M (of some dimension m) embeds into $\mathbb{R}^N$, for some $N \gg 0$.

   (c) **An introduction to knot theory**. A *knot* is an embedding of the compact 1-manifold S^1 into 3-dimensional Euclidean space $\mathbb{R}^3$. Roughly, speaking, the image of a knot is any closed loop you can draw in $\mathbb{R}^3$ which does not intersect itself. Any two knots are homeomorphic as topological spaces (equipped with the subspace topology from $\mathbb{R}^3$), as they are each homeomorphic to S^1. There is another notion of being "the same" that is relevant for knots (or more generally, any embeddings): that of *isotopy*.

   Roughly speaking, two knots K_0, K_1 are *isotopic* if I can continuously deform one loop to another without creating any self-crossings.[1] A remarkable fact is that there are many pairs of knots that are not isotopic, and to this date an exact classification of isotopy classes of knots is unknown! (*Knot theory* is an active area of research in mathematics that studies such questions).

   The goal of this project is to initiate the study of classifying *isotopy classes of knots*: define knots (you'll need to define what an "embedding" $S^1 \to \mathbb{R}^3$ is, but not more general types of embeddings) and isotopies of knots, give a brief survey of the classification question, and find with proof (though you may assume whatever fundamental facts about knot theory you wish)

(3) **Topological dimension of a space**: Using open-covers, define a number called the *topological dimension* of any topological space M. The goal of this project is to explore this definition (given in Munkres §8.50), prove some properties of dimension (the lemmas and Propositions in that section), and give examples (solve some of the exercises in §8.50).

(4) **Methods of constructing topological spaces**. Explore the notion of *CW complexes* (a good reference is Hatcher's Algebraic Topology book, Chapter 0), and/or *simplicial complexes* (and say what a *triangulation* of a topological space is). Give

[1] Formally, two knots $f : S^1 \to \mathbb{R}^3$ and $g : S^1 \to \mathbb{R}^3$ are *isotopic* if one can find a continuous family of embeddings f_t, $t \in [0,1]$ which equals f at $t = 0$ and g at $t = 1$; "continuous family of embeddings" means that (i) f_t should be an embedding for all $t \in [0,1]$, and (ii) (continuity) the map $S^1 \times [0,1] \to \mathbb{R}^3$ sending $(s,t) \mapsto f_t(s)$ is continuous (in both s and t variables).

many examples (the sphere, torus, higher genus surfaces, for instance); discuss properties of these spaces (e.g., when are they compact or connected?)

(5) **Space-filling curves**: We have proven in class that $\mathbb{R}$ and $\mathbb{R}^2$ are not homeomorphic; that is, there is no continuous function $\mathbb{R} \to \mathbb{R}^2$ with continuous inverse. Similarly, the line segment $[0, 1]$ (with its subspace topology) and the square $[0, 1]^2$ are not homeomorphic. [2]. On the other hand, when considered as sets, $[0, 1]$ and $[0, 1]^2$ (resp. $\mathbb{R}$ and $\mathbb{R}^2$) are in bijection (why? construct one bijection as part of this project). One can ask an intermediate question: is there is a *continuous* bijection $[0, 1] \to [0, 1]^2$? (necessarily, the inverse will not be continuous) Bizarrely, the answer is yes! Such maps are called *space-filling curves*; the first such curve was discovered by Peano in 1890.

The goal of this project is to give an overview of the above discussion and then construct, with proof, one such space filling curve $f : [0, 1] \to [0, 1]^2$, for instance the one detailed in Munkres §7.44. You will probably need to prove some theorems about *completeness of metric spaces*; the desired f is constructed by a limiting procedure; one constructs approximations f_n for each n and argues that a limit must exist (by *completeness*).

(6) **Invariants of topological spaces, and their applications**: An *invariant* of a topological space X is some sort of gadget one assigns to X, $k(X)$, which is (a) hopefully easy to compute (for instance, a number), and is (b) unchanged by homeomorphism. (meaning that if $X \cong Y$, then $k(X) = k(Y)$). Invariants give a tool for understanding whether spaces are homeomorphic; more precisely, they give us tools for understanding when spaces are *not* homeomorphic. (e.g., if $k(X) \neq k(Y)$, then X cannot be homeomorphic to Y).

At the end of semester, we will define one invariant of a topological space, called the *fundamental group* $\pi_1(X)$. $\pi_1(X)$ is a *group*[3], and it is an invariant: if $X \cong Y$, then $\pi_1(X) \cong \pi_1(Y)$.

The goal of this project, loosely defined, is to recall the definition, and study applications of, the fundamental group (or another invariant of your choice—examples include *Euler characteristic* and *Homology*). Here are some suggested applications, which can be found in the textbook; each could consist of a different project:

(a) **Homotopy equivalences**. Study the notion of a *deformation retract*, and more generally a *homotopy equivalence*, which are weaker notions than two spaces being homeomorphic (give examples). Prove that π_1 is unchanged under homotopy equivalences, and use this to compute π_1 in some new examples. Good references include Hatcher's Algebraic Topology book (Chapter 0, available online), or Munkres §9.58.

(b) **Fundamental theorem of algebra**. The fundamental theorem of algebra states that any polynomial $a_n z^n + \cdots + a_1 z + a_0$, with $a_i \in \mathbb{C}$ complex numbers, has a zero (in the complex numbers). It is a remarkable fact that this theorem can be proven using topology, specifically the fundamental group π_1. Give an

[2]The proof is the same as the case of $\mathbb{R}$ and $\mathbb{R}^2$; roughly note that if we remove the point $\frac{1}{2}$, $[0, 1] - \frac{1}{2}$ is no longer connected; on the other hand, $[0, 1]^2$ minus any point is always connected

[3]Recall that a *group* is a set G with a multiplication operation $\cdot : G \times G \to G$ satisfying some properties (consult references.

account of this proof, using a reference of your choice (one account is in Munkres §9.56).