Math 535a Homework 2

Due Monday, January 29, 2018 by 5 pm

Please remember to write down your name on your assignment.

1. (*double weight problem*) Let V be a vector space over a field k and let $W \subset V$ be a subspace (meaning a subset satisfying: for any $\mathbf{a}, \mathbf{b} \in W, c \cdot \mathbf{a} + d \cdot \mathbf{b} \in W$ for any scalars $c, d \in k$. Such a subset is automatically a vector space over k; with all operations coming from those in V).

   In class, we defined the quotient V/W to be the set-theoretic quotient of V by the equivalence relation $x \sim y$ iff $x - y \in W$. Equivalently, the quotient is the partition of V consisting of sets of the form $[\mathbf{a}] = \mathbf{a} + W = \{\mathbf{a} + w | w \in W\}$.

   (a) Show that V/W is a vector space, with operations induced by those of V in the following sense: for α and β in V/W, choose elements $\mathbf{a}$ and $\mathbf{b}$ with $[\mathbf{a}] = \alpha$, $[\mathbf{b}] = \beta$ and define $\alpha + \beta = [\mathbf{a} + \mathbf{b}]$ and $c \cdot \alpha = [c \cdot \mathbf{a}]$ (one needs to first show that these definitions, which require choosing representatives $\mathbf{a}$ and $\mathbf{b}$ of equivalence classes $\alpha = [\mathbf{a}]$, $\beta = [\mathbf{b}]$, is independent of choice, and then show that the axioms of being a vector space are satisfied).

   (b) The quotient comes equipped with a natural linear map

   $$\begin{aligned} \pi : V &\longrightarrow V/W \\ \mathbf{v} &\longmapsto [\mathbf{v}] = \mathbf{v} + W, \end{aligned}$$

   called the *projection,* which has $\ker \pi = W$ (check that π is linear and has kernel as desired). Suppose V is finite-dimensional, and let U be a subspace complementary to W, that is a subspace such that $U \cap W = \{\mathbf{0}\}$ and $V = W + U = W \oplus U$. Show that the restriction of projection to U

   $$\pi_U : U \longrightarrow V/W$$

   is an isomorphism.

   (c) Let $C^\infty(\mathbb{R})$ denote the vector space of *smooth* functions $f : \mathbb{R} \to \mathbb{R}$ (namely, $f \in C^\infty(\mathbb{R})$ if f' exists and is continuous, f'' exists and is continuous, and so on[1]. Examples include many of the standard functions you know—sin, cos, polynomials, exponentials, etc.)

   Let U denote the subspace of $C^\infty(\mathbb{R})$ consisting of functions which vanish at 3 and 5

   $$U = \{f \in C^\infty(\mathbb{R}) \mid f(3) = f(5) = 0\};$$

[1] You may take for granted this is a vector space, with operations $(f+g)(x) = f(x)+g(x)$ and $(c \cdot f)(x) = c \cdot (f(x))$; the key point is that these operations preserve the condition of being smooth

you may take for granted that U is a subspace.[2] Prove that the quotient vector space $C^\infty(\mathbb{R})/U$ is finite-dimensional. What is its dimension? In contrast, note that $C^\infty(\mathbb{R})$ is infinite dimensional.[3]

(d) Let V be a vector space and $W \subset V$ be a vector subspace. We denote by $i : W \to V$ the inclusion map. Recall we defined the dual of a vector space as $V^* = \mathrm{Hom}_k(V, k)$. There is a natural induced map $i^* : V^* \to W^*$ dual to the inclusion sending a linear map $\phi \mapsto \phi|_W$. The kernel of i^* is called the *annihilator* of W and denoted

$$\mathrm{Ann}(W) = \{\phi \in V^* | \phi|_W = \mathbf{0} \in W^*\}$$

it is the set of linear maps from V to k that return 0 on any element in W.

Prove that there is a canonical isomorphism

$$\mathrm{Ann}(W) \cong (V/W)^*$$

Hint: there is a natural map $\pi^* : (V/W)^* \to V^*$ dual to the projection $\pi : V \to V/W$. Describe this map and prove that π^* an isomorphism onto its image, which is $\mathrm{Ann}(W)$, or equivalently that π^* is injective and surjective onto $\mathrm{Ann}(W)$.

2. Let $S^n = \{(x_1, \ldots, x_{n+1}) | x_1^2 + \cdots + x_{n+1}^2 = 1\} \subset \mathbb{R}^{n+1}$. Prove that S^n has the structure of a smooth manifold, using charts associated to the cover $U_N = \{x_1 \neq +1\}$, $U_S = \{x_1 \neq -1\}$. (Hint: as in the case of S^1 in class, use *sterographic projection* to map U_N, respectively U_S to $\mathbb{R}^n$).

3. Prove that the product of two smooth manifolds $(M^m, \mathcal{A}_M = \{(U_\alpha, \phi_\alpha : U_\alpha \to \mathbb{R}^m)\}_{\alpha \in I})$, $(N, \mathcal{A}_N = \{(V_\beta, \psi\beta : V_\beta \to \mathbb{R}^m)\}_{\beta \in J})$ naturally has the structure of a smmooth manifold, with atlas given by $\mathcal{A}_{M \times N} = \{(U_\alpha \times V_\beta, (\phi_\alpha, \psi_\beta) : U_\alpha \times V_\beta \to \mathbb{R}^m \times \mathbb{R}^n = \mathbb{R}^{m+n})\}_{(\alpha,\beta) \in I \times J}$.

4. Prove that the antipodal map $S^n \to S^n$, $\mathbf{x} \mapsto -\mathbf{x}$ is a diffeomorphism of manifolds.

5. Finish the proof from class that $\mathbb{R}P^n$ is a smooth manifold (of dimension n).

6. Finish the proof from class that $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ is a smooth 2-manifold.

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[2]Proof: If $f, g \in U$, then note that $c \cdot f + d \cdot g \in U$ for any scalars c, d, because e.g., $(c \cdot f + d \cdot g)(3) = c \cdot f(3) + d \cdot g(3) = c \cdot 0 + d \cdot 0 = 0$ (and similarly when one evaluates at 5)

[3]Proof: Let $f_i = x^i \in C^\infty(\mathbb{R})$ for each $i \in \mathbb{Z}_{\geq 0}$. Suppose a finite collection $f_{i_1}, \ldots, f_{i_k}$ spanned $C^\infty(\mathbb{R})$; therefore a collection of the form $f_0, f_1, \ldots, f_N$ spans $C^\infty(\mathbb{R})$ for some $N = \max(i_1, \ldots, i_k)$. Now, note that the smooth function x^{N+1} is linearly independent from x^0, x^1, $\ldots$, x^{N+1}, a contradiction. (why is this the case?)