Math 535a Homework 2

Due Friday, February 25, 2022 by 5 pm

Please remember to write down your name on your assignment.

1. Let $U \subset \mathbb{R}^m$, $V \subset \mathbb{R}^n$ be open sets and $f : U \to V$ a smooth map, and $p \in U$ a point. Prove as stated in class that, under the natural identification $\mathbb{R}^m \cong T_pU$, $\mathbb{R}^n \cong T_{f(p)}V$ (sending in each case $\vec{e}_i$ to $\frac{\partial}{\partial x_i}$), the derivative df_p as constructed in class (using definition 2 of tangent space) coincides with, or reduces to, the usual derivative mapping of f at p, from $\mathbb{R}^m$ to $\mathbb{R}^n$ (as defined at the start of class).

2. Show that the two definitions of a submanifold $Y^m \subset N^n$ given in class are equivalent. Namely, show that Y is the image of an embedding $M^m \hookrightarrow N^n$ if and only if at every point $p \in Y$, there exists a chart (U, ϕ) in N's maximal atlas, containing (and centered at) p, such that $\phi(U \cap Y) = \phi(U) \cap \{x_{m+1} = x_{m+2} = \cdots = x_n = 0\} = \phi(U) \cap (\mathbb{R}^m \times \{0\})$.

3. Prove that $S^n = \{x_1^2 + \cdots + x_{n+1}^2 = 1\} \subset \mathbb{R}^{n+1}$ can be given the structure of an n-dimensional manifold by exhibiting it as the regular value of some map.

4. Let $O(n) = \{A \in M_n(\mathbb{R}) | AA^T = I\}$ be the *orthogonal group*, where A^T is the *transpose* of A. Consider the map

$$\begin{aligned} \phi : M_n(\mathbb{R}) &\to \mathrm{Sym}(n) \\ A &\mapsto AA^T \end{aligned}$$

   where $Sym(n) = \{B \in M_n(\mathbb{R}) | B = B^T\}$ is the set of *symmetric matrices*.

   (a) Show that $\mathrm{Sym}(n)$ is a submanifold of $M_n(\mathbb{R})$ (and in particular a manifold), and compute its dimension. **Hint**: It may be helpful to prove the following general Lemma: If V is a finite-dimensional vector space, it canonically has the structure of a smooth manifold (we proved this in class), and if $W \subset V$ is a linear subspace, then it is naturally a submanifold of V (to check).

   (b) Prove that $I \in \mathrm{Sym}(n)$ is a regular value of ϕ.

   (c) Prove that $O(n)$ is a submanifold of $M_n(\mathbb{R})$. What is its dimension?

   (d) Prove that $O(n)$ is compact.

5. Let Γ be a group and M a smooth manifold. A (C^∞) *action* of Γ on M is a group homorphism ρ from Γ to the group $\mathrm{Diff}(M)$ of diffeomorphisms on M. If $\gamma \in \Gamma$ and $x \in M$, we write $\gamma x = \rho(\gamma)(x)$ for the image of x under the diffeomorphism $\rho(\gamma)$.

   Recall from class that the *quotient space* M/Γ of the action Γ on M is the set of equivalence classes of the equivalence relation $\sim$ defined by $x \sim y$ iff $y = \gamma x$ for some

$\gamma \in \Gamma$.

(a) We say the action of Γ on M is *discontinuous* if, for every compact subset K of M, the set $\{\gamma \in \Gamma | K \cap \gamma K \neq \emptyset\}$ is finite. We say the action of Γ on M is *free* if $\gamma x \neq x$ for every $x \in M$ and $\gamma \in \Gamma - \{\text{id}\}$.

Prove that if Γ acts freely and discontinously on M, then the quotient M/Γ naturally has the structure of a smooth manifold. (this generalizes the manifold structure on $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ studied earlier)

(b) Let $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$ act on $S^n \subset \mathbb{R}^{n+1}$ by sending $x \mapsto -x$. Using the standard manifold structure on S^n (either as given above via expressing S^n as a preimage or as studied on last homework), prove that $S^n/\mathbb{Z}_2$ has the structure of a manifold, which is diffeomorphic to $\mathbb{R}P^n$, equipped with the smooth manifold structure which you defined on your last homework: (with charts $U_i = \{x_i \neq 0\}, \phi_i : U_i \mapsto \mathbb{R}^n$, $[x_0 : \cdots x_n] \mapsto (\frac{x_0}{x_i}, \frac{x_1}{x_i}, \cdots, \widehat{\frac{x_i}{x_i}}, \cdots, \frac{x_n}{x_i})$.

6. Let $M = \{(x, y, z) \in \mathbb{R}^3 | z = \sqrt{x^2 + y^2}\}$.
   (a) Show that $M - \{(0, 0, 0)\}$ is a 2-dimensional submanifold of $\mathbb{R}^3 - \{(0, 0, 0)\}$.

   (b) Let $\alpha : (-\epsilon, \epsilon) \to \mathbb{R}^3$ be a smooth curve with image contained in M, such that $\alpha(0) = (0, 0, 0)$. Show that $\alpha'(0) = (0, 0, 0)$. *Possible hint*: Write $\alpha(t) = (x(t), y(t), z(t))$, note that $z(t)^2 = x(t)^2 + y(t)^2$, and first prove that $z'(0) = 0$.

   (c) Use part (b) to show that M is not a submanifold of $\mathbb{R}^3$. *Hint*: otherwise, what would the tangent space $T_{(0,0,0)}M$ be?

7. (*2x weight*) Earlier in class, we defined the notion of a *category* $\mathcal{C}$; examples given include *topological spaces* **Top**, and *vector spaces* **Vect**.

   A *functor* $F : \mathcal{C} \to \mathcal{D}$ from category $\mathcal{C}$ to $\mathcal{D}$ is an assignment, to every object of $\mathcal{C}$, an object of $\mathcal{D}$, and an induced map on morphism spaces. More precisely, a *(covariant) functor* $F : \mathcal{C} \to \mathcal{D}$ is specified by the following data:
   - A map on object $F : \text{ob } \mathcal{C} \to \text{ob } \mathcal{D}$
   - For every pair of objects X, Y, a map on morphism spaces $F = F_{XY} : \hom_{\mathcal{C}}(X, Y) \to \hom_{\mathcal{D}}(F(X), F(Y))$, which satisfies:
     - F sends identity morphisms to identity morphisms (so $F(id_X) = id_{F(X)}$, where $X \in \text{ob } \mathcal{C}$.), and
     - F is compatible with compositions, in the sense that $F(g) \circ F(f) = F(g \circ f)$ for any objects X, Y, Z and morphisms $g \in \hom(Y, Z)$, $f \in \hom(X, Y)$.

   A *contravariant functor* from $\mathcal{C}$ to $\mathcal{D}$, written as

$$G : \mathcal{C}^{op} \to \mathcal{D},$$

consists of the following data: [1]

- A map on object $G : \text{ob } \mathcal{C} \to \text{ob } \mathcal{D}$
- For every pair of objects X, Y, a map on morphism spaces $G = G_{XY} : \hom_{\mathcal{C}}(X, Y) \to \hom_{\mathcal{D}}(G(Y), G(X))$ (note the order reversal), which satisfies:
  - G sends identity morphisms to identity morphisms (so $G(id_X) = id_{G(X)}$, where $X \in \text{ob } \mathcal{C}$.), and
  - G is compatible with compositions, in the sense that $G(f) \circ G(g) = G(g \circ f)$ for any objects X, Y, Z and morphisms $g \in \hom(Y, Z)$, $f \in \hom(X, Y)$.

In other words, a contravariant functor is specified by the same sort of data as a covariant functor, except the order of morphisms in the target is reversed in passing from the source to the target category.

(a) To any topological space M, define a category $\mathbf{Open}(M)$ as follows:

- objects of $\mathbf{Open}(M)$ are the open subsets $U \subset M$.
- Morphisms from U to V are *inclusions*, meaning that: if U is not contained in V, then $\hom(U, V) = \emptyset$, and if $U \subset V$, then $\hom(U, V) = \{i_{UV} : U \hookrightarrow V\}$, where i_{UV} simply denotes the inclusion map $U \hookrightarrow V$.
- Composition of morphisms $\hom(V, W) \times \hom(U, V) \to \hom(U, W)$ (which is only non-trivial if $U \subset V \subset W$) is the usual composition of inclusions. Namely $i_{VW} \circ i_{UV} = i_{UW}$.

Verify that $\mathbf{Open}(M)$ satisfies the axioms of a category.

(b) A **pre-sheaf** on M taking values in a category $\mathcal{C}$ is a contravariant functor

$$F : \mathbf{Open}(M)^{op} \to \mathcal{C}.$$

For instance, if $\mathbf{Alg}_{\mathbb{R}}$ denotes the category of $\mathbb{R}$-algebras (objects are $\mathbb{R}$ algebras,[2] and morphisms are $\mathbb{R}$-algebra homomorphisms[3], then a *pre-sheaf of* $\mathbb{R}$*-algebras* on M is a functor $F : \mathbf{Open}(M) \to \mathbf{Alg}_{\mathbb{R}}$.

Verify that the notion of a pre-sheaf of algebras $\mathcal{F}$ is equivalent to the following data:

- For every open set $U \in M$, an algebra $\mathcal{F}(U)$.

- For every inclusion of open sets $U \subseteq V$, a restriction map $\rho_{U \subset V} : \mathcal{F}(V) \to \mathcal{F}(U)$, satisfying, $\rho_{U \subset U} = id_{\mathcal{F}(U)}$, and for any triple $U \subset V \subset W$, that $\rho_{U \subset V} \circ \rho_{V \subset W} = \rho_{U \subset W}$.

[1] A contravariant functor from $\mathcal{C}$ to $\mathcal{D}$ is the same as a covariant functor from the *opposite category* $\mathcal{C}^{op}$ of $\mathcal{C}$ to $\mathcal{D}$, hence the notation. We will not elaborate on this point more here.

[2] Let k be any field. For our purposes, a k*-algebra* A is a vector space over k equipped with a multiplication map $\times : A \times A \to A$ which is a bilinear map. We further assume that the mutiplication map is associative, and that there is a multiplicative identity $1 \in A$ satisfying $1 \cdot \alpha = \alpha \cdot 1 = \alpha$, for $\alpha \in A$ (elsewhere, such A are frequently called *associative unital algebras*). You should verify for yourself that if U is any manifold, then $C^\infty(U)$ is an $\mathbb{R}$-algebra in this sense.

[3] For our purposes, an k*-algebra homomorphism* $F : A \to B$ is a linear map of vector spaces which is compatible with the multiplication maps, meaning that $F(\alpha \cdot \beta) = F(\alpha) \cdot F(\beta)$. F should also preserve the identity elements, so $F(1) = 1$; this is frequently elsewhere called a *unital algebra homomorphism.* You should verify for yourself that if $f : M \to N$ is any C^∞ map, then the pullback $f^* : C^\infty(N) \to C^\infty(M)$ is an $\mathbb{R}$-algebra homomorphism

(c) Let M be a smooth manifold now. Show that there is a pre-sheaf of algebras on M $C^\infty(-)$, i.e., a contravariant functor $C^\infty(-) : \mathbf{Open}(M)^{op} \to \mathbf{Alg}_\mathbb{R}$ which sends an an open subset $U \subset M$ to
$$U \mapsto C^\infty(U),$$
and sends a morphism i.e., an inclusion $i_{UV} : U \to V$ to the induced $\mathbb{R}$-algebra morphism $C^\infty(-)_{UV}(i_{UV}) \in \hom_{\mathbf{Alg}_\mathbb{R}}(C^\infty(V), C^\infty(U))$ given by restriction on functions $C^\infty(-)_{UV}(i_{UV}) = i^*_{UV} : C^\infty(V) \to C^\infty(U)$.
(e.g., verify this satisfies the conditions of a pre-sheaf/contravariant functor).

(d) A pre-sheaf as defined in the previous section is said to be a *sheaf* if for any pair of open sets U, V, whenever there is an element $f_1 \in \mathcal{F}(U)$ and an element $f_2 \in \mathcal{F}(V)$ with the same restriction on the overlapping region,[4] then there *exists* a *unique* element $g \in \mathcal{F}(U \cup V)$ restricting to f_1 and f_2 on U and V.[5]

Let M be a manifold. Verify that the pre-sheaf on M, $C^\infty(-)$ defined above is in fact a sheaf.

(e) *another example of a functor*: Let $\mathbf{Man}^+$ denote the category of *pointed manifolds*, defined as follows: objects are pairs (M, p) of a manifold M and a point $p \in M$, and the set of morphisms $\hom((M, p), (N, q))$ consist of the set of smooth maps from M to N taking p to q, with composition the usual composition of maps. Verify that $\mathbf{Man}^+$ is a category and prove that the assignment sending an object (M, p) to the vector space T_pM and a morphism $f : (M, p) \to (N, q)$ to the linear map $df_p : T_pM \to T_qN$ defines a functor $D : \mathbf{Man}^+ \to \mathbf{Vect}$.

8. Let $M = f^{-1}(y)$ be the preimage of a regular value $y \in \mathbb{R}^{N-m}$ of a smooth function $f : \mathbb{R}^N \to \mathbb{R}^{N-m}$. (for instance, $M = S^2 = \{x^2 + y^2 + z^2 = 1\} \subset \mathbb{R}^3 = f^{-1}(1)$, where $f : (x, y, z) \mapsto x^2 + y^2 + z^2$).

(a) Let $\widetilde{TM} = \{(x, v) \in \mathbb{R}^N \times \mathbb{R}^N | x \in M, v \in \ker df_x\}$. Show that as defined, $\widetilde{TM}$ is a smooth submanifold of $\mathbb{R}^N \times \mathbb{R}^N$ of dimension $2m$ (where M is an m-dimensional manifold).

(b) Prove that there is a diffeomorphism between $\widetilde{TM}$ and the *tangent bundle of* M as defined in class:
$$\widetilde{TM} \cong TM$$
compatible with the natural projection to M on both sides. (It follows that, for instance, $TS^2 \cong \{(x, v) \in \mathbb{R}^3 \times \mathbb{R}^3 | x \in S^2 | v \cdot x = 0\}$.

Note: this is a strengthening of the fact stated in class that for such an f and for any such $p \in M$, $T_pM = \ker(df_p)$. More generally, similar methods can show that $\{(p, v)\ p \in M, v \in \ker(df_p)\}$ is a smooth manifold diffeomorphic to $M = f^{-1}(y)$ for any smooth $f : N \to Q$ and any regular value $y \in Q$.

[4]meaning that $\rho_{U\cap V \subset U}(f_1) = \rho_{U \cap V \subset V}(f_2)$
[5]meaning that $\rho_{U \subset U \cup V}(g) = f_1$, $\rho_{U \subset U \cup V}(g) = f_2$.