Math 535a Homework 3

Due Wednesday, April 20, 2022 by 5 pm

Please remember to write down your name on your assignment.

For the linear algebra problems, recall the following definition of *matrix associated to a linear operator with respect to a basis*: If V is a finite-dimensional vector space, $T : V \to V$ a linear operator, and $\underline{v} = k\{v_1, \dots, v_n\}$ a basis of V, then we say that T *has matrix A, with respect to $\underline{v}$*, and write

$$\mathcal{M}(T, \underline{v}) = A = (A_{ij})_{(i,j)=(1,1)}^{(n,n)} = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & \cdots & \cdots & A_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{pmatrix}$$

(where A_{ij} denotes entry at *row i, column j*) if, for each basis element v_s, the vector $T(v_s)$ can be expressed in terms of the basis $\underline{v}$ as $T(v_s) = \sum_{i=1}^n A_{is} v_i$.

Recall that a linear map is determined by what it does to any basis; the coefficients of the matrix A then track what scalar weights are necessary to describe each $T(v_i)$ as a linear combination of elements of $\underline{v}$.

Remark. *For maps $T : \mathbb{R}^n \to \mathbb{R}^n$, with respect to the standard basis $\mathcal{S} = \{e_1, \dots, e_n\}$, we get the usual notion of matrix. That is, given an $n \times n$ matrix A, with associated linear map $T_A : \mathbb{R}^n \to \mathbb{R}^n$ sending $v \mapsto Av$; we have $\mathcal{M}(T_A, \mathcal{S}) = A$ (indeed note that $T_A(e_s) = Ae_s = \sum_{i=1}^n A_{is} e_i$).*

1. Give a detailed proof that the cotangent bundle T^*M is a smooth manifold and that the projection map $\pi : T^*M \to M$ is a smooth map.

2. Let $i : S^1 = [0, 2\pi]/(0 \sim 2\pi) \to \mathbb{R}^2$ be the map $\theta \mapsto (\cos(\theta), \sin(\theta))$. Compute $i^*((x^2 + y)dx + (3 + xy^2)dy)$.[1]

3. **Linear algebra of tensor products**
   (a) Write in detail the construction of the canonical map $V^* \otimes W \xrightarrow{\alpha} \hom(V, W)$, and give a careful proof that it is an isomorphism if V and W are finite dimensional (in class, we constructed the map sketchily, as the linear map associated to a given bilinear map $V^* \times W \to \hom(V, W)$). *Hint*: Let $\underline{v} = (v_1, \dots, v_k)$ be a basis for V and $\underline{w} = (w_1, \dots, w_r)$ a basis for W. There is an associated *dual basis* for V^* given by $\underline{v}^* = (v_1^*, \dots, v_k^*)$, where v_j^* is the linear map $V \to \mathbb{R}$ that sends $v_j \mapsto 1$ and $v_i \mapsto 0$ if $i \neq j$. Write down an associated basis of $V^* \otimes W$ and check that α maps it to an associated basis of $\hom(V, W)$.

[1] As discussed in class, the notation $f_1 dx + f_2 dy$, where f_1 and f_2 are smooth functions on $\mathbb{R}^2$, is a common shorthand for the 1-form $\mathbb{R}^2 \to T\mathbb{R}^2 = \mathbb{R}^2 \times \mathbb{R}^2$ sending $\vec{x}$ to $(\vec{x}, (f_1(\vec{x})dx + f_2(\vec{x})dy))$.

(b) Let $ev : V^* \otimes V \to \mathbb{R}$ be the linear map induced by the bilinear map $\bar{ev} : V^* \times V \to \mathbb{R}$, $(\phi, v) \mapsto \phi(v)$ by the universal property of tensor product.[2] Given a linear operator $T \in \hom(V, V)$ on a finite-dimensional vector space, define the *trace* of T as

$$tr(T) := ev(\alpha^{-1}(T)),$$

where α is the map defined in the previous section.

Show that this definition agrees with the usual notion of trace, that is if $\underline{v} := \{v_1, \dots, v_k\}$ is any basis of V and A is the matrix of T with respect to $\underline{v}$, then $tr(T) = tr(A) = \sum_{i=1}^n a_{ii}$.

4. **Exterior algebra 1.**

(a) *A formula for the determinant of* 3×3 *matrices.* Recall from class that the determinant $\det(T)$ of $T \in \hom_{\mathbb{R}}(V, V)$ is defined as the scalar in $\mathbb{R}$ such that

$$T(v_1) \wedge \cdots \wedge T(v_n) = \det(T) \cdot v_1 \wedge \cdots \wedge v_n,$$

where $v_1, \dots, v_n$ is any basis for V.

Suppose that $\dim V = 3$, and $\underline{v} = (v_1, v_2, v_3)$ is a basis for V. Let $T : V \to V$ be the linear operator defined by

$$\begin{aligned} T(v_1) &= av_1 + dv_2 + gv_3 \\ T(v_2) &= bv_1 + ev_2 + hv_3 \\ T(v_3) &= cv_1 + fv_2 + iv_3. \end{aligned}$$

In other words, suppose the matrix of T with respect to $\underline{v}$ is

$$\mathcal{M}(T, \underline{v}) = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

Derive, using the definition we gave in class with exterior products, a formula for $\det(T)$ in terms of a, b, c, d, e, f, h, and i. (Remark: this formula should have six terms. It is easy to find the formula online, but you should be able to derive it in terms of the definition of determinants given in lecture).

5. **Exterior algebra 2.** For the below problems, let V be a finite-dimensional vector space over $\mathbb{R}$.

(a) Let $A^k(V) := \mathrm{AltMultiLin}_{\mathbb{R}}(\underbrace{V \times \cdots \times V}_{k \text{ times}}, \mathbb{R})$ be the vector space of alternating multilinear maps from k copies of V to $\mathbb{R}$ (what is its vector space structure). Also let $L^k(V) := \mathrm{MultiLin}_{\mathbb{R}}(\underbrace{V \times \cdots \times V}_{k \text{ times}}, \mathbb{R})$ is the vector space of multilinear maps.

Prove that there are canonical isomorphisms $A^k(V) \cong \wedge^k V^* \cong (\wedge^k V)^*$. Similarly, prove that there is a canonical isomorphism $L^k(V) \cong (V^*)^{\otimes k} \cong (V^{\otimes k})^*$, and that under

[2]We call ev and $\bar{ev}$ *evaluation maps*, hence the abbreviation.

these inclusions, the natural map $A^k(V) \hookrightarrow L^k(V)$ is sent to the (dual of) the projection map $(V)^{\otimes k} \to \wedge^k V$. (here we are implicitly using the fact that $(V^{\otimes k})^* \cong (V^*)^{\otimes k}$ and similarly for the wedge product).

(b) An element $\omega \in A^2(V) = \wedge^2 V^*$ is called *non-degenerate*, or a *linear symplectic form*, if $\omega(v, -) \neq 0 \in \hom_{\mathbb{R}}(V, \mathbb{R})$ for any non-zero $v \in V$. If V is finite-dimensional and V admits a linear symplectic form, prove that $n = \dim V$ is necessarily even, say $n = 2m$.

(c) Prove that $\omega \in \wedge^2 V^*$ is non-degenerate if and only if $\omega^m \neq 0$ in $\wedge^n V^*$ (where $n = 2m$). *Hint*: One possible method is to prove that ω is non-degenerate if and only if it has a nice form with respect to some basis of $\wedge^2 V^*$ (induced by a basis of V and hence V^*). Then using this form (with respect to some basis), calculate ω^m.

6. Give a careful construction of the exterior differentiation operator $d : \Omega^k(M) \to \Omega^{k+1}(M)$ using local coordinates; show that this definition is independent of local coordinates and is well-defined.

7. Let M be a manifold. Prove that d as defined in the previous problem satisfies the formula $d(\alpha \wedge \beta) = d(\alpha) \wedge \beta + (-1)^k \alpha \wedge d(\beta)$, where $\alpha \in \Omega^k(M)$, $\beta \in \Omega^l(M)$, including the case one or both of k, l equal 0 (where recall that for a 0-form i.e., a function f and a k form α, $f \wedge \alpha = \alpha \wedge f = f\alpha$). In particular show that $d : \Omega^k(M) \to \Omega^{k+1}(M)$ is *not* a map of $C^\infty(M)$-modules.

8. Prove that d commutes with pullback; that is, the following diagram commutes for any smooth $f : M \to N$ and for any $k \geq 0$:

$$\begin{array}{ccc} \Omega^k(N) & \xrightarrow{f^*} & \Omega^k(M) \\ \downarrow d & & \downarrow d \\ \Omega^{k+1}(N) & \xrightarrow{f^*} & \Omega^{k+1}(M) \end{array}$$