Math 535a : Differential Geometry I (Spring 2022)

(Topology)

Course mechanics:

$\boxed{\text{website}}$: https://sheelganatra.com/math535a/

Zoom links + course notes on Blackboard. (submit HW on Gradescope)

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Office Hours: To be regularized next week; this week: Thurs 4:30-6pm. (by Zoom)

Class time: Currently MWF 11am - 11:50am.

However, debating a switch to MW 11am - 12:20pm OR MW 10:30am - 11:50am. (w/ a backup time of F 11am - 12:20pm OR F 10:30 - 11:50)

Any conflicts?

(longer classes are a better format for this material, also will allow us the flexibility of having a backup time Friday we can use if we need | ake a class up).

Grading scheme:
- Homework 50%. ← roughly (slightly less than) weekly
- Midterm (takehome): 20%.
- Final exam (in class): 30%.

Textbook: Lee, Introduction to Smooth Manifolds (+ possibly other auxiliary references). ← available online through USC libraries / Springer Link

What is the course about?

"locally Euclidean spaces"

Goal: study (smooth, or differentiable) manifolds, and calculus on them (e.g., vector fields, differential forms, differentiation and integration, Stokes' theorem)

Examples of manifolds: $\mathbb{R}^n$, $S^n = \{x_1^2 + \cdots + x_{n+1}^2 = 1\} \subseteq \mathbb{R}^{n+1}$,

"manifold-with-boundary"

$S^2 =$ 

 "corners"

Prerequisites: Topology, linear algebra, differential & integral calculus on $\mathbb{R}^n$ (including basic theorems in ODEs)

(we'll review these as needed – w/o proofs – including for most of this week).

Ref: Appendix A–D of Lee's book

Topological spaces

Recall Def: A topological space is a pair $(X, \mathcal{T})$ where $\mathcal{T}$ satisfies:

set (X); collection of subsets of X, called a 'topology' on X ($\mathcal{T}$)

(1) ϕ and $X \in \mathcal{T}$

(2) Any finite intersection of elements of $\mathcal{T}$ again lies in $\mathcal{T}$.
(if $\{U_i\}_{i=1}^k$ all lie in $\mathcal{T}$ then $\bigcap_{i=1}^k U_i \in \mathcal{T}$).

(3) Arbitrary unions of elements of $\mathcal{T}$ again lie in $\mathcal{T}$.

The elements $U \in \mathcal{T}$ are called the open sets in $(X, \mathcal{T})$ (abuse of notation: $X := (X, \mathcal{T})$)

Also, call a set $V \subseteq X$ closed if $V^c := X - V$ is open.

Ex: Any set S has at least two topologies:

- trivial topology: $\mathcal{T} = \{\phi, S\}$.
- discrete topology: $\mathcal{T} = \{\text{all subsets of } S\}$

Ex: Let (X,d) be a metric space, i.e., a set X w/ $d: X \times X \longrightarrow [0, \infty)$ satisfying

(1) $d(x,y) = 0$ iff $x = y$ (otherwise $d > 0$)

(2) $d(x,y) = d(y,x)$ for all x, y.

(3) For any x, y, z, $d(x,z) \le d(x,y) + d(y,z)$

Given $x \in X := (X,d)$, $\varepsilon > 0$, the open ball of radius ε centered at x is:

$$B_\varepsilon(x) := \{ y \in X \mid d(y,x) < \varepsilon \}.$$

We get an induced topology $\mathcal{T} := \mathcal{T}_d$ as follows (the "metric topology" on X):

we say a subset $U \subseteq X$ is open (& hence in $\mathcal{T}$) if for any $x \in U$, there exists some ε with $B_\varepsilon(x) \subseteq U$.

"the topology generated by $\{B_\varepsilon(x)\}_{x\in X, \varepsilon>0}$" :

Note: $(\mathbb{R}^n, d(x,y) = \|x-y\| = \sqrt{(x_1-y_1)^2 + \cdots + (x_n-y_n)^2})$ is a metric space, hence a topological space.

Recall: • A top. space $X := (X, \mathcal{T})$ is Hausdorff if $\forall\ x,y \in X$, $x \neq y$, there exists a neighborhood U of x and V of y, so that $U \cap V = \varnothing$.

("open subset containing")

• A subset $A \subseteq X$ is dense in X if every non-empty open set in X contains an element of A.

(e.g., $\mathbb{Q}^n \subseteq \mathbb{R}^n$ is dense).

A top. space X is separable if it has a countable dense subset.

$\Rightarrow \mathbb{R}^n$ is separable.

Ways to construct topological spaces:

(a) X, Y top. spaces $\Rightarrow X \times Y$ is naturally a topological space with open sets = (arbitrary unions of) products of open sets.

"topology generated by products of open sets"

(b) (subspace topology):

Say $Y := (Y, \mathcal{T})$ is a top. space, and $i: X \hookrightarrow Y$ injective map.

(e.g., X subset & i inclusion)

Then X carries an induced topology "subspace topology" :

(if X subset, i inclusion, $i^{-1}(U_Y) = U_Y \cap X$)

say $U \subseteq X$ is open in X if it's of the form $i^{-1}(U_Y)$ where $U_Y \subseteq Y$ open.

(recall $f: A \to B$, $B \supseteq C$, $f^{-1}(C) = \{a \in A \mid f(a) \in C\}$)

Ex: $X = S^1 = \{x^2+y^2=1\} \subseteq \mathbb{R}^2$ ← top. space.

$\Rightarrow$ X inherits structure of a top. space. via subspace topology.

(c) (quotient topologies) Say $X := (X, \mathcal{T}_X)$ top. space and $p : X \twoheadrightarrow Y$ surjective map. (some set)

Then Y inherits a topology, the <u>quotient topology</u>, from X and p:

say $U \subseteq Y$ is open iff $p^{-1}(U) \subseteq X$ is open.

i.e., $\mathcal{T}_{\text{quotient}} := \{U \subseteq Y \mid p^{-1}(U) \in \mathcal{T}_X\}$.

<u>e.g.</u>: $[0,1] \subseteq \mathbb{R}$ is a top. space (by (b)) and there's a surjective map

$$[0,1] \xrightarrow{p} S^1 = \{z \in \mathbb{C} \mid |z| = 1\},$$
$$t \longmapsto \exp(2\pi i t).$$

giving S^1 the structure of a top. space via <u>quotient topology</u>.

<u>Continuous functions & homeomorphisms</u>

<u>Def</u>: X, Y top. spaces, $f : X \to Y$ is continuous if for every open $U \subseteq Y$, $f^{-1}(U)$ is open in X.

(for metric spaces (X, d_X), (Y, d_Y), this is equivalent to the ε-δ notion of continuity)

<u>Def</u>: A <u>homeomorphism</u> $f : X \to Y$ is a continuous map with a continuous two-sided inverse $g : Y \to X$.

For a homeomorphism f: $U \subseteq X$ is open iff $f(U) \subseteq Y$ is open.

Returning to above examples: one can show that

S^1 (w/ subspace topology from $\mathbb{R}^2$) is homeomorphic to S^1 (w/ quotient topology from $[0,1]$), and e.g., to

$[0,1] / 0 \sim 1$. with its quotient topology,

so these all give the 'same' topological space.