Topological manifolds

- course time MWF 11am for now
- OH tomorrow 4 – 5/5:30pm, via Zoom. (Zoom link on Blackboard, please email.)

Def: A top. space $M := (M, \mathcal{T})$ is called an <u>n-dimensional topological manifold</u> if

(1) M is Hausdorff

(2) M is <u>second countable</u> ⟵ **Def:** Y is <u>second countable</u> if it has a countable <u>basis</u> for its topology.

↖ A collection of open sets $\mathcal{B}$ is a <u>basis</u> if every open is a union of elements of $\mathcal{B}$.

(n.b., any <u>separable metric space</u> is second countable: if $A \subseteq (X,d)$ is countable dense, consider $\mathcal{B} = \{B_\varepsilon(a)\}_{a \in A,\, \varepsilon \in \mathbb{Q}_{>0}}$.

(e.g., $\mathbb{R}^n$ is second <u>countable</u>).

(3) M is locally Euclidean of dimension n: at every $x \in M$, $\exists$ an (open) nbhd U of x with $U \underset{\text{homeo.}}{\cong}$ an open subset of $\mathbb{R}^n$.

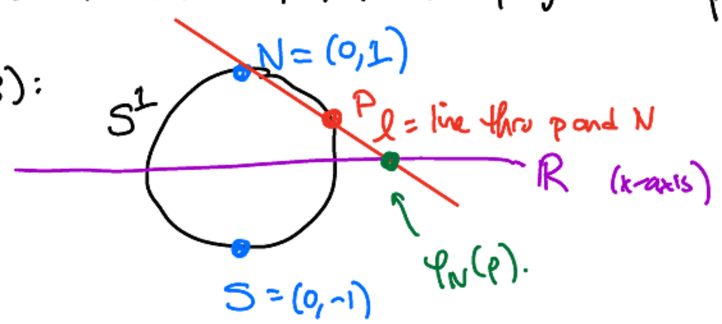
(<u>Equivalently</u>: $\exists$ an open cover $\{U_i\}_{i \in I}$ of M s.t. each $U_i \underset{\text{homeo.}}{\cong}$ an open subset of $\mathbb{R}^n$.).

<u>N.B.</u>, any such M has a well-defined (& fixed) dimension, which is a topological invariant:

<u>Thm</u>: A top. m-m'fold M and n-m'fold N cannot be homeomorphic unless $m = n$.

<u>Examples</u>: (see Lee Ch. 1 for more)

- $\mathbb{R}^n$ is a topological manifold (of dimension n).
- $S^1 \subseteq \mathbb{R}^2$ is a topological manifold (of dimension 1) (& more generally, $S^n \subseteq \mathbb{R}^{n+1}$).
  - properties (1) + (2) are inherited from $\mathbb{R}^2$ via passage to subspace
  - <u>property</u> (3):



Define $U_N = S^1 \setminus N$ (note: $\{U_S, U_N\}$ cover S^1).

$U_S = S^1 \setminus S$,

and $\varphi_N : U_N \longrightarrow \mathbb{R}$

$p \longmapsto$ the unique point on the x-axis $= \mathbb{R}$ which intersects ℓ, the line thru N and p.

and $\varphi_S : U_S \longrightarrow \mathbb{R}$ (same way as φ_N, with S instead of N)

check: • $\varphi_N((x,y)) = \frac{x}{1-y}$, $\varphi_S(x,y) = \frac{x}{1+y}$ ($\Rightarrow$ continuous)

• both φ_N & φ_S bijective

• can write inverses as

$$\varphi_N^{-1}(x) = \left(\frac{2x}{x^2+1}, \frac{x^2-1}{x^2+1}\right), \quad \varphi_S^{-1}(x) = \left(\frac{2x}{x^2+1}, \frac{1-x^2}{x^2+1}\right).$$

$\Rightarrow$ φ_S, φ_N continuous w/ continuous inverses, i.e., homeomorphisms to $\mathbb{R}$.

Note: on $S^1 \setminus \{S, N\}$, both φ_S and φ_N are defined & map to $\mathbb{R} \setminus \{0\}$, yet they don't agree: check that $\varphi_N \circ \varphi_S^{-1} = \frac{1}{x}$ & (vice versa)
(as functions $\mathbb{R}\setminus 0 \to \mathbb{R}\setminus 0$).

Review of linear algebra & calculus (beginning)

Linear algebra: A vector space over a field $k = \mathbb{R}$ or $\mathbb{C}$ is a set V equipped with operations

• "$+$" : $V \times V \to V$ (addition)

• "$\cdot$" : $k \times V \longrightarrow V$ (scalar multpl.)

satisfying:

(1) $(V, +)$ is an abelian group (in particular, $\exists$ additive identity $0 \in V$).

(2) • $1\vec{v} = \vec{v}$, • $a(\vec{v}+\vec{w}) = a\vec{v} + a\vec{w}$.

• $a(b\vec{w}) = (ab)\vec{w}$ • $(a+b)\vec{w} = a\vec{w} + b\vec{w}$. $\forall\, a, b \in k$, $\vec{v}, \vec{w} \in V$.

($\Rightarrow 0 \cdot \vec{v} = 0$, ...)

Def: A linear map $\phi : V \to W$ of vector spaces (over k) is a map of sets satisfying:

• $\phi(\vec{v}_1 + \vec{v}_2) = \phi(\vec{v}_1) + \phi(\vec{v}_2)$

• $\phi(c\vec{v}) = c\,\phi(\vec{v})$ (for any $c \in k$, $\vec{v}, \vec{v}_1, \vec{v}_2 \in V$).

Such a ϕ is an isomorphism if there exists a linear map $\psi: W \to V$ with
$\psi \circ \phi = id_V \quad \phi \circ \psi = id_W$.

$\exists$ finite spanning set

If V, W are finite-dimensional, can choose bases $\{\vec{v}_1, \ldots, \vec{v}_k\}$ of V, $\{\vec{w}_1, \ldots, \vec{w}_\ell\}$ of W
and represent a linear map $\phi: V \to W$ by a matrix A.

Constructing vector spaces:

(0) $\mathbb{R}^k$ is a vector space (over $k = \mathbb{R}$) with usual $\cdot, +$

(1) Given V, W, define $\mathrm{Hom}_k(V, W) = \{\phi: V \to W \text{ linear maps}\}$. This is again a vector space.

(a vector space over k)

In particular, when $W = k$, write $V^* = \mathrm{Hom}_k(V, k)$ dual vector space.
(or $V^\vee$)

(2) $\phi: V \to W$ induces $\ker(\phi) \subseteq V$, $\ker(\phi) = \{v \,/\, \phi(v) = 0\}$
$\mathrm{im}(\phi) \subseteq W$, $\mathrm{im}(\phi) = \{w \,/\, \exists\, v \text{ with } w = \phi(v)\}$.

(3) $V \subset W$ subspace. Then the quotient vector space $W/V = \{w + V \,/\, w \in W\}$ is a vector space.

($\Rightarrow$ $\phi: V \to W$ also induces $\mathrm{coker}(\phi) = W/\mathrm{im}(\phi)$).

Rmk: A category $\mathcal{C}$ is the following collection of data:

- a collection (class) of objects $\mathrm{ob}\,\mathcal{C}$.
- For each pair of objects $X, Y \in \mathrm{ob}\,\mathcal{C}$ a set/collection of morphisms $\hom_{\mathcal{C}}(X, Y)$
- A composition rule, for any triple of objects X, Y, Z

$$- \circ - : \hom_{\mathcal{C}}(Y, Z) \times \hom_{\mathcal{C}}(X, Y) \longrightarrow \hom_{\mathcal{C}}(X, Z)$$

satisfying certain conditions:
- composition is associative: $f \circ (g \circ h) = (f \circ g) \circ h$

- $\exists$ elements $id_X \in hom_{\mathcal{C}}(X,X)$ for any $X \in ob\,\mathcal{C}$ which "behave like identity with respect to compositions," i.e.,

  $f \circ id_X = f = id_Y \circ f$ for any X, Y, $f \in hom_{\mathcal{C}}(X,Y)$.

  <u>identity elements</u>

So far, we've already implicitly seen two categories:

(1) $\mathcal{C} = \mathbf{Top}$: • objects $ob\,\mathbf{Top} = \{$topological spaces$\}$.

- For X, Y topological spaces, $hom_{Top}(X,Y) = \{$continuous maps $f: X \to Y\}$

$$= C(X,Y)$$

"continuous functions"

<u>check</u>: • $id_X : X \to X$ is continuous, thus $id_X \in hom_{Top}(X,X)$

- usual composition satisfies the axioms of a category.

(2) $\mathcal{C} = Vect_k$: • objects are vector spaces (over k).

- For V, W vector spaces, $Hom_{Vect}(V,W) = Hom_k(V,W)$ linear maps.

(note: $Hom_{Vect}(V,W)$ is not just a set, it's a vector space again — this is a special property of $Vect_k$).

In a category $\mathcal{C}$, a morphism $f \in hom_{\mathcal{C}}(X,Y)$, written $f: X \to Y$ is called an <u>isomorphism</u> if $\exists$ $g: Y \to X$ with $g \circ f = id_X$ $f \circ g = id_Y$.

Note that:

- in Top, the isomorphisms are homeomorphisms.
- in $Vect_k$, the isomorphisms are linear isomorphisms (/invertible linear maps).

<u>Next time</u>: reviewing multivariable differential calculus in $\mathbb{R}^n$; smoothness, smooth manifolds.