• No class Monday, resume Wednesday (by Zoom)

• HW 1 coming soon.

Differentiation review (ref. Lee, Appendix C).

Let $f: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ (not necessarily linear)

or, $U \subseteq \mathbb{R}^n$ open, $V \subseteq \mathbb{R}^m$, and $f: U \to V$.

Def: Say such an f is differentiable at $\vec{a} \in \mathbb{R}^n$ (or $\vec{a} \in U$) if there exists a linear map $L: \mathbb{R}^n \to \mathbb{R}^m$ satisfying:

$$\lim_{\substack{\vec{h} \to 0 \\ (\vec{h} \in \mathbb{R}^n - \vec{0})}} \frac{|f(\vec{a}+\vec{h}) - f(\vec{a}) - L(\vec{h})|}{|\vec{h}|} = 0.$$

($L(\vec{h}) \sim f(\vec{a}+\vec{h}) - f(\vec{a})$ with error $o(|\vec{h}|^2)$ as $\vec{h} \to 0$).

If exists, L is called the derivative of f at $\vec{a}$ and written $df_{\vec{a}}$ or $df(\vec{a})$ (or $T_{\vec{a}} f$ "tangent map")

Exercise: show if L exists it's unique.

If f is differentiable at x then get $df(x): \mathbb{R}^n \to \mathbb{R}^m$ is a linear map, defined by

$$df(x)(\vec{v}) = \lim_{t \to 0} \frac{f(x+t\vec{v}) - f(x)}{t}.$$

← call this (if exists) the directional derivative of f at x in direction v.

differentiability ⇒ all directional derivatives exist.

Particular directional derivatives are given special importance: Let $\vec{e}_i = (0, \ldots, 0, \underset{i\text{th}}{1}, 0, \ldots 0)$.

If $g: \mathbb{R}^n \longrightarrow \mathbb{R}$, then define

$$\frac{\partial g}{\partial x_i}(a) = \lim_{t \to 0} \frac{g(a+t\vec{e}_i) - g(a)}{t} \quad \text{(if exists)}$$

and for $f: \mathbb{R}^n \longrightarrow \mathbb{R}^m$, write $f = (f_1, \ldots, f_m)$, define:

$$\underbrace{\frac{\partial f}{\partial x_i}(a)}_{\partial_i f(a)} = \begin{bmatrix} \frac{\partial f_1}{\partial x_i}(a) \\ \vdots \\ \frac{\partial f_m}{\partial x_i}(a) \end{bmatrix} \quad \text{(if it exists)}$$

Prop: If all partial derivatives of f exist near a and are continuous at a, then f is differentiable at a, and moreover $df(a)$ has matrix

$$\begin{bmatrix} \frac{\partial f_1}{\partial x_1}(a) & \frac{\partial f_1}{\partial x_2}(a) & \cdots & \frac{\partial f_1}{\partial x_n}(a) \\ \vdots & \vdots & \cdots & \vdots \\ \frac{\partial f_m}{\partial x_1}(a) & \frac{\partial f_m}{\partial x_2}(a) & \cdots & \frac{\partial f_m}{\partial x_n}(a) \end{bmatrix}$$

(w.r.t. the standard basis).

This matrix is sometimes called the Jacobian matrix.

Fact: If $\partial_i(\partial_j f)$ and $\partial_j(\partial_i f)$ (exist and) are continuous near a, then $\partial_i(\partial_j f) = \partial_j(\partial_i f)$.

We say: f is

note: class C^0 means simply continuous.

- k-differentiable at x (or of class C^k) if each k^{th} partial derivative exists in a nbhd of x and is continuous at x.

  (k-differentiable over $U \subseteq \mathbb{R}^n \iff$ k-diff. at every point in U).

- smooth (or of class C^∞) if it is infinitely differentiable (i.e., in C^k $\forall k$)

  (i.e., all derivatives exist and are continuous.)

Assuming f is smooth, write $\partial^{\vec{v}} f = \partial_1^{v_1} \partial_2^{v_2} \cdots \partial_n^{v_n} f$ where $\vec{v} = (v_1, \ldots, v_n)$ (each $v_i \in \mathbb{Z}_{\geq 0}$.)

Chain rule: Let $f: \mathbb{R}^n \to \mathbb{R}^m$ differentiable at $a \in \mathbb{R}^n$, $g: \mathbb{R}^m \to \mathbb{R}^p$ differentiable at $f(a) \in \mathbb{R}^m$.

(or: $f: U \to V$, $g: V \to W$ where $U \underset{\text{open}}{\subseteq} \mathbb{R}^n$, $V \underset{\text{open}}{\subseteq} \mathbb{R}^m$, $W \underset{\text{open}}{\subseteq} \mathbb{R}^p$).

Then $g \circ f$ is differentiable at a, and moreover

$$d(g \circ f)(a) = dg(f(a)) \circ df(a)$$

composition of linear transformations.

$U \subseteq \mathbb{R}^n$ open, $V \subseteq \mathbb{R}^m$ open.

Def: A map $f: U \to V$ is a C^∞ diffeomorphism if f is a smooth map (C^∞) with smooth inverse $f^{-1}: V \to U$

(have also a notion of a C^k diffeo., note a "C^0 diffeo." is just a homeomorphism)

(Note: (check) $\exists$ a cat. Cart^∞ whose objects are $U \underset{\text{open}}{\subseteq} \mathbb{R}^n$ & morphisms are C^∞ maps; C^∞ diffeos. are isomorphisms in this category).

Prop: If $f: U \to V$ is a (smooth) diffeomorphism, then $df(x): \mathbb{R}^n \to \mathbb{R}^m$ is an isomorphism for all $x \in U$. (in particular, if U non-empty then n must equal m).

Proof: Let $g: V \to U$ be the (smooth) inverse of f; it's smooth and

$g \circ f = id_U$; $f \circ g = id_V$.

Take derivatives & apply the chain rule (& note e.g., $d(id_U)(x) = id_{\mathbb{R}^n}$); this establishes that the linear map $df(x)$ has inverse $dg(f(x))$ for all $x \in U$. ☒.

There is a partial converse, which is very important:

Thm: (Inverse function theorem (multivariable calculus version, Lee Thm C.34))

Let $f: U \to V$ be a C^∞ map. Suppose that at $x \in U$, $df(x): \mathbb{R}^n \to \mathbb{R}^m$ is an isomorphism, (so in particular $n = m$).

($U \underset{\text{open}}{\subseteq} \mathbb{R}^n$, $V \underset{\text{open}}{\subseteq} \mathbb{R}^m$)

Then: There exists an (open) nbhd $U' \subseteq U$ of x such that $f(U')$ is open in $V \subseteq \mathbb{R}^m$ and the restriction $f|_{U'}: U' \to f(U')$ is a diffeomorphism.

See Lee for a proof.

Manifolds

Topological manifold restatement:

Def: A top. manifold of dim. n. is a second countable, Hausdorff, space X s.t. $\exists$ an open cover $\{U_\alpha\}_{\alpha \in I}$ of X and a collection of maps $\mathcal{A} := \{\phi_\alpha : U_\alpha \to \mathbb{R}^n\}_{\alpha \in I}$ satisfying:

- ϕ_α is a homeo. onto an open subset $\phi_\alpha(U_\alpha)$ of $\mathbb{R}^n$.

("called a chart" ; "called an atlas")

Exercise: Show that $\square$ in $\mathbb{R}^2$ is a top. manifold, and give an example of a second countable Hausdorff space which is not a top. manifold.

Differentiable (smooth) manifolds:

A smooth manifold (or differentiable, or C^∞) is a top. n-manifold equipped with a fixed choice of

$$\mathcal{A} = \{\phi_\alpha : U_\alpha \to \mathbb{R}^n\} \quad (\text{where } U_\alpha \underset{\text{open}}{\subseteq} X \ \& \ \{U_\alpha\}_{\alpha \in I} \text{ covers } X)$$

satisfying:

for every U_α, U_β with $U_\alpha \cap U_\beta \neq \emptyset$, the maps

$$\phi_\beta \circ \phi_\alpha^{-1} : \underset{\substack{\cap \text{ open} \\ \mathbb{R}^n}}{\phi_\alpha(U_\alpha \cap U_\beta)} \longrightarrow \underset{\substack{\cap \\ \mathbb{R}^n.}}{\phi_\beta(U_\alpha \cap U_\beta)}$$

"transition map"

is a smooth map.