• Office hours this week at Thursday 4pm–5pm, Friday 3pm–4pm

• HW 1 coming soon.

Recall:

<u>Def</u>: A <u>top. manifold</u> of dim. n. is a (second countable, Hausdorff (**)) space X s.t. $\exists$ an open cover $\{U_\alpha\}_{\alpha\in I}$ of X and a collection of maps $\mathcal{A} := \{\phi_\alpha : U_\alpha \to \mathbb{R}^n\}_{\alpha \in I}$ ("atlas"; ϕ_α "chart") satisfying:

- ϕ_α is a homeo. onto an open subset $\phi_\alpha(U_\alpha)$ of $\mathbb{R}^n$.

(a top. manifold is an X satisfying (**) s.t. $\exists$ "topological atlas" $\mathcal{A}$ as above.

<u>Def</u>: A <u>smooth</u> (or C^∞, or differentiable manifold) of dimension n is a pair of a topological manifold equipped with a choice of atlas $(X, \mathcal{A})$ satisfying the following condition:

- for every U_α, U_β in $\mathcal{A}$ with $U_\alpha \cap U_\beta \neq \emptyset$, the <u>transition maps</u>

$$\underset{\substack{\cap\text{ open}\\ \mathbb{R}^n}}{\phi_\alpha(U_\alpha \cap U_\beta)} \xrightarrow{\phi_\beta \circ \phi_\alpha^{-1}} \underset{\substack{\cap\text{ open}\\ \mathbb{R}^n}}{\phi_\beta(U_\alpha \cap U_\beta)}$$

are <u>C^∞ maps</u>. (Since $\phi_\alpha \circ \phi_\beta^{-1}$ is also therefore C^∞ & inverse to $\phi_\beta \circ \phi_\alpha^{-1}$, it follows that the transition maps must be <u>diffeomorphisms</u>.)

- each ϕ_α is called a <u>chart</u>.
- $\phi_\beta \circ \phi_\alpha^{-1}$: <u>transition map</u>.

Going forward: <u>manifold</u> := smooth manifold.

<u>Examples of (smooth) manifolds</u>:

(1) $\mathbb{R}^n$ is a smooth manifold, with atlas

$$\mathcal{A} := \{ U_1 = \mathbb{R}^n \xrightarrow{\phi_1 = id} \mathbb{R}^n \}$$ only one chart.

$\mathcal{A}_1 = \{U_1 \xrightarrow{\phi_1 = id} \mathbb{R}^n\}$ $\mathcal{A}_2 = \{U_2 \xrightarrow{f_{non\text{-}diff}} \mathbb{R}^n\}$ are each smooth atlases.

<u>Note</u>: Pick your favorite $f_{non\text{-}diff} : \mathbb{R}^n \underset{\text{homeo.}}{\xrightarrow{\cong}} \mathbb{R}^n$ which is <u>not</u> C^∞.

Then consider the topological atlas on $\mathbb{R}^n$ given by $\{U_1 = \mathbb{R}^n \xrightarrow{\phi_1 = id} \mathbb{R}^n,\ U_2 = \mathbb{R}^n \xrightarrow{\phi_2 = f_{non\text{-}diff}} \mathbb{R}^n\}$

Note $\phi_2 \circ \phi_1^{-1} = f_{non\text{-}diff} : \mathbb{R}^n \to \mathbb{R}^n$, which is <u>not</u> smooth.

(1′): V n-dim'l vector space over $\mathbb{R}$, fix a linear isomorphism $T: V \xrightarrow{\cong} \mathbb{R}^n$ (equivalently, a basis), $\Rightarrow$ get an atlas $\mathcal{A}_T := \{V \underset{\cong}{\xrightarrow{T}} \mathbb{R}^n\}$

Later: any two $\mathcal{A}_T$'s are "equivalent."

(2) Any open subset U of a smooth manifold M is again a smooth manifold.

Given an atlas $\mathcal{A} = \{\phi_\alpha : U_\alpha \to \mathbb{R}^n\}$ for M, an atlas for U is

$$\mathcal{A}|_U = \{\phi_\alpha|_{U_\alpha \cap U} : U_\alpha \cap U \longrightarrow \mathbb{R}^n\}.$$ (check: $(U, \mathcal{A}|_U)$ is a smooth manifold).

sub.ex: $M_{n\times n}(\mathbb{R}) = \{n \times n \text{ matrices}\} \cong \mathbb{R}^{n^2}$ is an n^2-dim'l manifold by (1), &

$GL_n(\mathbb{R}) = \{n\times n$ matrices A with $\det(A) \neq 0\} \underset{\text{open}}{\subseteq} M_{n\times n}(\mathbb{R})$

$\Rightarrow$ $GL_n(\mathbb{R})$ is a manifold of dimension n^2.

Rmk: $(X, \mathcal{A} = \{U_\alpha \xrightarrow{\phi_\alpha} \mathbb{R}^n\})$ manifold, and $f: Y \to X$ homeo, then $(Y, f^*\mathcal{A} = \{f^{-1}(U_\alpha) \xrightarrow{\phi_\alpha \circ f} \mathbb{R}^n\})$ manifold too.

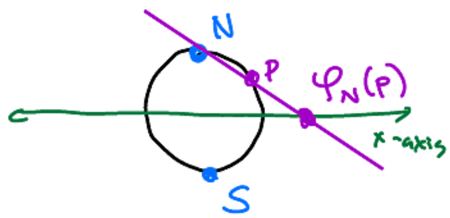
(3) M^m m-dim'l manifold, N^n n-dim'l manifold $\Rightarrow$ $M \times N$ is a (smooth) manifold of dimension $m+n$.

Give atlases $\mathcal{A}_M = \{\phi_\alpha : U_\alpha \to \mathbb{R}^m\}$ of M

$\mathcal{A}_N = \{\psi_\beta : V_\beta \to \mathbb{R}^n\}$ of N, get an atlas $\{\phi_\alpha \times \psi_\beta : U_\alpha \times V_\beta \to \mathbb{R}^m \times \mathbb{R}^n \cong \mathbb{R}^{m+n}\}$

(4) $S^1 = \{x^2 + y^2 = 1\} \subseteq \mathbb{R}^2$ is a smooth manifold (rather, S^1 can be given the structure of a smooth manifold)

- one possible atlas: the atlas from last time:



$U_N = S^1 \setminus N$, $U_S = S^1 \setminus S$,

$\varphi_N : U_N \longrightarrow \mathbb{R}$ $\quad p \longmapsto$ the point on x-axis lying on line through N and p.

$\varphi_S : U_S \longrightarrow \mathbb{R}$

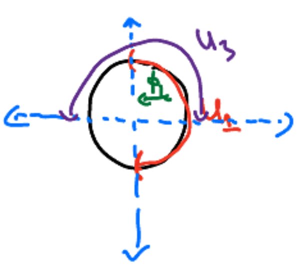
(last time: wrote explicit equations, & left as an exercise the fact that

$$\varphi_N \circ \varphi_S^{-1} : \varphi_S(U_N \cap U_S) = \mathbb{R}\setminus 0 \xrightarrow{\frac{1}{x}} \varphi_N(U_N \cup U_S) = \mathbb{R}\setminus 0,$$

which is a C^∞ map, as is $\varphi_S \circ \varphi_N^{-1} = \frac{1}{x}$

$\Rightarrow$ S^1, $\mathcal{A} = \{\varphi_N, \varphi_S\}$ is a smooth manifold.

- another possible atlas: (thinking of $S^1 \subseteq \mathbb{R}^2_{x,y}$):



$U_1 = \{x > 0\} \xrightarrow{\phi_1 = \text{projection to } y\text{-axis}} \mathbb{R}.$

$(x,y) \longmapsto y.$

$U_2 = \{x < 0\} \xrightarrow{\phi_2 = \text{project to } y\text{-axis}} \mathbb{R}.$

$$U_3 = \{y>0\} \xrightarrow{\phi_3 = \text{proj. to } x\text{-axis}} \mathbb{R}$$

$$U_4 = \{y<0\} \xrightarrow{\phi_4 = \text{proj. to } x\text{-axis}} \mathbb{R}.$$

Check: U_1, U_2, U_3, U_4 cover S^1, & each ϕ_i is a homeo. onto its image in $\mathbb{R}$ (image is always $(-1,1)$).

Is $(S^1, \mathcal{A} = \{\phi_1, \phi_2, \phi_3, \phi_4\})$ a smooth manifold? Exercise: show it is. Start:

e.g., on $U_1 \cap U_3 = \{x>0, y>0\}$

$$\underset{(0,1)}{\underset{\shortparallel}{\phi_1(U_1 \cap U_3)}} \xrightarrow{\phi_3 \circ \phi_1^{-1}} \underset{(0,1)}{\underset{\shortparallel}{\phi_3(U_1 \cap U_3)}}$$

$$t \longmapsto (+\sqrt{1-t^2}, t) \longmapsto +\sqrt{1-t^2}.$$ smooth on $(0,1)$ ✓.

(need to check all pairwise overlapping intersections)

Next time: In some sense the two atlases above for S^1, $\mathcal{A} = \{\varphi_N, \varphi_S\}$, $\mathcal{A}' = \{\phi_1, \dots, \phi_4\}$ give the "same" smooth manifold.

(5) Exercise: Construct a smooth manifold structure (i.e., a smooth atlas) on $S^n = \{\sum_{i=1}^{n+1} x_i^2 = 1\} \subseteq \mathbb{R}^{n+1}$ by generalizing either of above atlases.

(6) (overlapping): $(n=2)$: S^2, $T^2 = S^1 \times S^1$, ... genus g surfaces

"genus 0" "genus 1" Σ_g has "g holes".

(7) $\mathbb{RP}^n$ real projective space "manifold of lines in $\mathbb{R}^{n+1}$" (generalization: "Grassmannian of k-dim'l subspaces of $\mathbb{R}^{n+k}$")

- as a space, $\mathbb{RP}^n = (\mathbb{R}^{n+1} \setminus (0, \dots, 0))/\sim$

where $(x_0, \dots, x_n) \sim (tx_0, \dots, tx_n)$ for any $t \in \mathbb{R} \setminus 0$.

Notation: on $\mathbb{R}P^n$, let $[x_0 : x_1 \cdots : x_n]$ denote an equivalence class of $(x_0, \ldots, x_n)$ under relation above.

So e.g., $[1:2:3] = [\frac{1}{2}, 1, \frac{3}{2}]$

Next time: smooth manifold structure on $\mathbb{R}P^n$, more examples, "equivalent smooth structures".