Examples of (smooth) manifolds, continued from last time

HW 1 coming today, due next week.

(7) $\mathbb{RP}^n$ real projective space "manifold of lines in $\mathbb{R}^{n+1}$"

- as a space, $\mathbb{RP}^n = \left(\mathbb{R}^{n+1} \setminus (0,\dots,0)\right)/\sim$

where $(x_0,\dots,x_n) \sim (tx_0,\dots,tx_n)$ for any $t \in \mathbb{R}\setminus 0$.

Notation: on $\mathbb{RP}^n$, let $[x_0 : x_1 \cdots : x_n]$ denote an equivalence class of $(x_0,\dots,x_n)$ under relation above.

So e.g., $[1:2:3] = [\tfrac{1}{2}, 1, \tfrac{3}{2}]$ (in $\mathbb{RP}^2$)

On $\mathbb{RP}^n$, define $U_i = \left(\{x_i \neq 0\} - \{\vec{0}\} \subseteq \mathbb{R}^{n+1} \setminus \vec{0}\right)/\sim$ for $i = 0,\dots,n$.

$[x_0 : \cdots : x_n] \in U_i$ iff for any rep. $(x_0,\dots,x_n)$, $x_i \neq 0$,

and a map

$$\phi_i : U_i \longrightarrow \mathbb{R}^n$$

$$[x_0 : \cdots : x_n] = \Big[\tfrac{x_0}{x_i} : \tfrac{x_1}{x_i} : \cdots : \underset{i^{th}}{1} : \tfrac{x_{i+1}}{x_i} : \cdots : \tfrac{x_n}{x_i}\Big] \longmapsto \left(\tfrac{x_0}{x_i}, \tfrac{x_1}{x_i}, \dots, \tfrac{x_{i-1}}{x_i}, \tfrac{x_{i+1}}{x_i}, \dots, \tfrac{x_n}{x_i}\right).$$

(i.e., for $x \in U_i$, $\phi_i(x)$ has components equal to all but the ith component of the unique representative of x with i^{th} component equal 1).

Note $\{U_i\}_{i=0}^n$ covers $\mathbb{RP}^n$ & each $\phi_i : U_i \underset{\text{homeo.}}{\xrightarrow{\cong}} \mathbb{R}^n$ (check).

Exercise: check using $\mathcal{A} = \{ U_i \xrightarrow{\phi_i} \mathbb{R}^n \}$, $\mathbb{RP}^n$ becomes a smooth manifold.

(8) Quotients by "nice" group actions:

Another way to define the torus is by

$$T^2 = \mathbb{R}^2 / \mathbb{Z}^2$$

The discrete group $\mathbb{Z}^2$ acts on $\mathbb{R}^2$ by translation:

$$\mathbb{Z}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

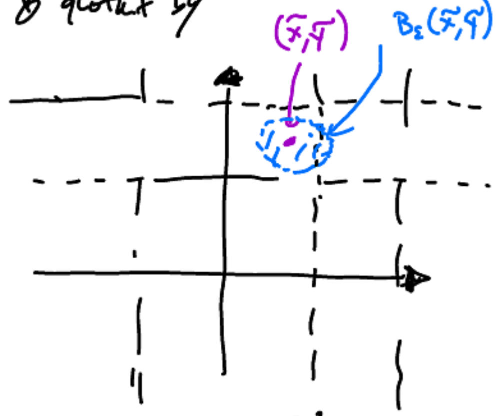
$$((m,n),(x,y)) \longmapsto (m+x,\ n+y)$$

$\mathbb{R}^2/\mathbb{Z}^2$ is the set of orbits of $\mathbb{R}^2$ under the action of $\mathbb{Z}^2$,

e.g., one orbit is the set $(x,y)+\mathbb{Z}^2$.

Equivalently, can take a "fundamental domain" $[0,1]\times[0,1]$ & quotient by

$(0,y)\sim(1,y)$ and $(x,0)\sim(x,1)$.



Can define n-torus $T^n = \mathbb{R}^n/\mathbb{Z}^n$ identically.

charts: At any point $p \in T^2$, pick a representative $(\tilde{x},\tilde{y}) \in [(x,y)]$ in $\mathbb{R}^2$.

$[(x,y)] = (x,y)+\mathbb{Z}^2$

and pick $\varepsilon \ll 1$. Then note that the map

$$\begin{tikzcd} B_\varepsilon(\tilde{x},\tilde{y}) \arrow[r, hook] \arrow[rr, bend right, "f"'] & \mathbb{R}^2 \arrow[r, "\text{proj.}"] & \mathbb{R}^2/\mathbb{Z}^2 \end{tikzcd}.$$

is, for ε small, a homeo. onto its image.
(in particular its injective

Let U_p denote the image of $B_\varepsilon(\tilde{x},\tilde{y})$, & use as a chart

$$\phi_p = f^{-1} : U_p \xrightarrow{\cong} B_\varepsilon(\tilde{x},\tilde{y}) \underset{\text{open}}{\subseteq} \mathbb{R}^2.$$

Atlas: For each $p\in T^2$, pick a rep'n $(\tilde{x},\tilde{y})$ and a small ε $\Rightarrow$ get (U_p,ϕ_p).

Check: These charts give T^2 (or more generally T^n) the structure of a smooth manifold.

Note: In case $n=1$, get yet another smooth atlas on $S^1 \underset{\text{homeo.}}{\cong} T^1 = \mathbb{R}/\mathbb{Z}$ or $[0,1]/0\sim1$.

General question: Do above atlases on S^1, for instance, give the "same" manifold?

Choice of atlas: Let $M := (M,\mathcal{T})$ be underlying top. space, and

$$\mathcal{A}_1 = \{U_\alpha, \phi_\alpha : U_\alpha \to \mathbb{R}^n\}_{\alpha\in I},\quad \mathcal{A}_2 = \{V_\beta, \psi_\beta : V_\beta \to \mathbb{R}^n\}_{\beta\in J}$$ two

smooth atlases. When do they represent the "same" manifold?

(gives an equivalence relation on smooth atlases)

Def: Two smooth atlases $\mathcal{A}_1$ and $\mathcal{A}_2$ on M are <u>compatible</u> if the union $\mathcal{A}_1 \cup \mathcal{A}_2$ is still a <u>smooth</u> atlas, or equivalently that for every $(U_\alpha, \phi_\alpha) \in \mathcal{A}_1$, $(V_\beta, \psi_\beta) \in \mathcal{A}_2$ with $U_\alpha \cap V_\beta \neq \emptyset$ the transition map

$$\underset{\substack{\cap \text{ open} \\ \mathbb{R}^n}}{\phi_\alpha(U_\alpha \cap V_\beta)} \xrightarrow[\psi_\beta \circ \phi_\alpha^{-1}]{} \underset{\substack{\cap \text{ open} \\ \mathbb{R}^n}}{\psi_\beta(U_\alpha \cap V_\beta)}$$

is a smooth map. (as is $\phi_\alpha \circ \psi_\beta^{-1}$).

Note: If $\mathcal{A}_1, \mathcal{A}_2$ compatible, by defn $\mathcal{A}_1 \cup \mathcal{A}_2$ is a (smooth) atlas compatible with both $\mathcal{A}_1$ & $\mathcal{A}_2$.

Def: Given a smooth manifold $(M, \mathcal{A})$, its <u>maximal atlas</u> $\mathcal{A}_{max} = \{(U_\alpha, \phi_\alpha)\}$ is an atlas which is compatible with $\mathcal{A}$ and contains every atlas $\mathcal{A}' \supseteq \mathcal{A}$ ~~on~~ which is compatible with $\mathcal{A}$. (in particular, $\mathcal{A}_{max}$ contains every atlas compatible with $\mathcal{A}$)

Exercises (apply Zorn's lemma): Every atlas $\mathcal{A}$ of M is contained in a unique maximal atlas $\mathcal{A}_{max}$ of M.
(Also $\Rightarrow$ if $\mathcal{A}, \mathcal{A}'$ are compatible then $(\mathcal{A})_{max} = (\mathcal{A}')_{max}$)

Call the maximal atlas $\mathcal{A}_{max}$ of $(M, \mathcal{A})$ the <u>differentiable structure</u> on M.

Say that $(M, \mathcal{A}_1)$ and $(M, \mathcal{A}_2)$ are the "same" smooth manifold if they have the same <u>diff. structure</u> i.e., if $(\mathcal{A}_1)_{max} = (\mathcal{A}_2)_{max}$ ($\Leftrightarrow \mathcal{A}_1, \mathcal{A}_2$ compatible).

Might write $\mathcal{A}_1 \sim^{C^\infty} \mathcal{A}_2$.

<u>Functions</u>: For a top. space, the space of functions we've studied is

$$C^0(X) := \text{continuous fcns } f: X \to \mathbb{R}, = \hom_{Top}(X, \mathbb{R})$$

$$\text{or more generally } C^0(X, Y) := \hom_{Top}(X, Y)$$

<u>Idea</u>: Smooth (C^∞) manifolds have well-defined notions of smooth (C^∞) functions (either just to $\mathbb{R}$ or between two smooth manifolds more generally).

Functions to $\mathbb{R}$ first:

Fix $\left(M, \mathcal{A} = \{(U_\alpha, \phi_\alpha : U_\alpha \to \mathbb{R}^n)\}\right)$ smooth manifold.

Def: A function $f : M \to \mathbb{R}$ is smooth (or C^∞) at a point $p \in M$ if $\exists\, U_\alpha \ni p$ so that

$$f \circ \phi_\alpha^{-1} : \underset{\underset{\phi_\alpha(p)}{\rotatebox{90}{$\in$}}}{\overset{\overset{\mathbb{R}^n}{\cup\,\text{open}}}{\phi_\alpha(U_\alpha)}} \longrightarrow \mathbb{R} \quad \text{is smooth at } \phi_\alpha(p).$$

Lemma: The above definition is $\Longleftrightarrow$ the condition that for any $U_\beta \ni p$, the map

$$f \circ \phi_\beta^{-1} : \phi_\beta(U_\beta) \longrightarrow \mathbb{R} \quad \text{is smooth at } \phi_\beta(p).$$

Pf: $\Longleftarrow$ is immediate.

$\Longrightarrow$ Say found (U_α, ϕ_α) s.t. $f \circ \phi_\alpha^{-1}$ is smooth at $\phi_\alpha(p)$.

And let (U_β, ϕ_β) another chart with $U_\beta \ni p$.

Observe that $U_\alpha \cap U_\beta \neq \emptyset$ because contains p, and

$$\begin{tikzcd}
 & \overset{\phi_\alpha(p).}{\phi_\alpha(U_\alpha \cap U_\beta)} \arrow[dr, "f \circ \phi_\alpha^{-1}"] \arrow[d, phantom, "\circlearrowright"] & \\
\phi_\beta(U_\alpha \cap U_\beta) \arrow[ur, "\phi_\alpha \circ \phi_\beta^{-1}"] \arrow[rr, "f \circ \phi_\beta^{-1}"'] & & \mathbb{R}.
\end{tikzcd}$$

$\left(\text{meaning } (f \circ \phi_\alpha^{-1}) \circ (\phi_\alpha \circ \phi_\beta^{-1}) = f \circ \phi_\beta^{-1}\right).$

smoothness of transition fncts is key

Now $f \circ \phi_\alpha^{-1}$ smooth at $\phi_\alpha(p)$, $\phi_\alpha \circ \phi_\beta^{-1}$ smooth everywhere and sends $\phi_\beta(p)$ to $\phi_\alpha(p)$.

$\Rightarrow$ the composition $f \circ \phi_\beta^{-1}$ is smooth at $\phi_\beta(p)$.

Def: $f : (M, \mathcal{A}) \to \mathbb{R}$ is <u>smooth</u> if it's smooth at every point.

Next time: this notion only depends on the differentiable structure $\mathcal{A}_{max}$ of $\mathcal{A}$.